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Duality method in limit analysis problem of non-linear elasticity

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The limit analysis problem (LAP) for estimation of mechanical durability for non-linear elastic solids is examined. The appropriate dual problem is formulated. After the standard piecewise linear continuous finite-element approximation, the dual LAP is transformed into the problem of mathematical programming with linear limitations as equalities. This finite dimensional problem is solved by the standard method of gradient projection.

Keywords: non-linear elastic solid, limit analysis problem, duality method

1. Introduction

Investigation of the elastostatic boundary-value problems (BVPs) is of particular interest in both theory and practice. It is stimulated by significance and practical interest in Mechanical Engineering.

In this paper the finite elasticity BVP is formulated as the variational problem for the displacement. The porous rubber-like materials working in water or oil are described by elastic potentials having the linear growth in the modulus of the distortion tensor [1]. For such materials, the existence of the limiting static load (external static forces with no solution of the BVP), and discontinuous maps with jumps of the sliding type were proved by the author in [2–9]. From the physical point of view these effects are treated as the *global* destruction of a solid [7, 8].

The limit analysis problem for non-linear elastic solids is examined [7, 8]. Within the framework of this problem the estimation from below for the limiting static load is calculated. But from the mathematical point of view this problem needs a relaxation because its solution belongs to the space $BV(\Omega, \mathbb{R}^3)$ of vector functions with bounded variations having the generalized gradient as the bounded Radon's measure [10, 11]. The original partial relaxation of the limit analysis problem was proposed in [7, 8]. This relaxation is based on the special discontinuous finite-element approximation (FEA) which also has been adapted by the author recently for LAP in electrostatics for dielectrics in powerful electrical fields [12].

Using methods of duality theory, the *dual* LAP in finite elasticity is formulated. From the mathematical point of view this problem is fully correct and after the standard piecewise linear continuous FEA it is transformed into the problem of mathematical programming with linear limitations as equalities. This finite dimensional problem is effectively solved by the standard method of gradient projection which is easily adapted for *parallel calculations*.

The numerical results show that the proposed technique has the qualitative advantage over standard methods of estimation of the global mechanical durability for non-linear elastic solids. This technique is very relevant to Mechanical Engineering.

2. LAP IN FINITE ELASTICITY

Let a body in the undeformed reference configuration occupy a domain $\Omega \subset \mathbb{R}^3$. In the deformed configuration each point $x \in \overline{\Omega}$ moves into the position $X(x) = u(x) + x \in \mathbb{R}^3$, where X and u are the map and displacement, respectively. Here and in what follows Lagrangian's co-ordinates are used. We consider locally invertible and orientation-preserving maps $X : \overline{\Omega} \to \mathbb{R}^3$ with gradient (*the distortion tensor*) $Q(X) = \nabla X : \Omega \to \mathbb{M}^3$ such that $\det(Q) > 0$ in Ω [13, 14], where the symbol $\mathbb{M}^3$ denotes the space of real 3×3 matrices

The finite deformation of materials is described by the energy pair (Q, Σ), where $\Sigma = \{\Sigma_i^\alpha\}$ is the first non-symmetrical Piola–Kirchhoff stress tensor [13, 14]. It is known that the Cauchy stress tensor σ has the components $\sigma^{\alpha\beta} = (\det(Q))^{-1}\Sigma_i^\alpha Q_i^\beta$. Here and in what follows the Roman lower and Greek upper indices correspond to the reference and deformed configurations, respectively, and the rule of summing over repeated indices is used.

Elastic materials are characterized by the scalar function (*elastic potential*) $\Phi : \Omega \times \mathbb{M}^3 \to \mathbb{R}_+$ such that $\Sigma_i^\alpha = \partial\Phi(x, Q)/\partial Q_i^\alpha$ for every $Q \in \mathbb{M}^3$ and almost every $x \in \Omega$, and $\Phi(x, I) = 0$, where $I = \mathrm{Diag}(1, 1, 1)$. If a material is incompressible, then $\det(Q) = 1$, but for a compressible material $\Phi(x, Q) \to +\infty$ as $\det(Q) \to +0$.

We consider the following boundary-value problem. The quasi-static influences acting on the body are: a mass force with density f in Ω, a surface force with density F on a portion Γ^2 of the boundary, and a portion Γ^1 of the boundary is fixed, i.e. $u \equiv 0$ on Γ^1. Here $\Gamma^1 \cup \Gamma^2 = \partial\Omega$, $\Gamma^1 \cap \Gamma^2 = \emptyset$ and $|\Gamma^1| > 0$.

For hyperelastic materials the finite elasticity BVP is formulated as the following variational problem [2–9, 13, 14]:

$$u^* = \arg\inf\{I(u) : \quad u \in V\}, \tag{1}$$

$$I(u) = \int_\Omega \Phi(x, \nabla u(x) + I)\, dx - A(u),$$

$$A(u) = \int_\Omega \langle f, u\rangle\, dx + \int_{\Gamma^2} \langle F, u\rangle\, d\gamma, \qquad \langle g, u\rangle(x) = \int_0^{u(x)} g^\alpha(x, v)\, dv^\alpha.$$

Here $V = \{u : \overline{\Omega} \to \mathbb{R}^3; \quad u(x) = 0, \quad x \in \Gamma^1\}$ is the set of kinematically admissible displacements, $\langle *, u\rangle$ is the specific and $A(u)$ is the full work of the external forces under the displacement u. It must be noted that even for "dead" forces, i.e. $f, F = \mathrm{const}(u, \nabla u)$, the specific work has the form of $\langle g, u\rangle(x) = g^\alpha(x)u^\alpha(x)$ only in Descartes's coordinates [3, 4].

Definition 1. *An elastic material has the ideal saturation if a positive constant $C > 0$ exists such that for every matrix $Q \in \mathbb{M}^3$ with $|\mathrm{Cof}Q| \le C$ and $C^{-1} \le \det(Q) \le C$, the elastic potential satisfies the following estimation from above:*

$$\Phi(x, Q) \le C|Q - I|.$$

Here and in what follows $|Q| = (Q_i^\alpha Q_i^\alpha)^{1/2}$.

According to the general theory [15], for potentials of linear growth the set of admissible displacements is a subspace of the non-reflexive Sobolev's space $W^{1,1}(\Omega, \mathbb{R}^3)$

$$V = \{u \in W^{1,1}(\Omega, \mathbb{R}^3) : \quad u(x) = 0, \quad x \in \Gamma^1\}. \tag{2}$$

We remind the definition of the limiting static load [2]. For this reason we introduce the set of admissible "dead" external forces for which the functional $I(u)$ is bounded from below on V and, therefore, a solution of the problem (1) exists:

$$B = \{(f, F) \in L^\infty(\Omega, \mathbb{R}^3) \times L^\infty(\Gamma^2, \mathbb{R}^3) : \quad \inf\{I(u) : u \in V\} > -\infty\}.$$

This set is non-empty because for small external influences the problem (1) is transformed into the classical variational problem of linear elasticity [14] which always has a solution [13].

For arbitrary external forces $(f_0, F_0) \in B$ we examine the sequence of forces which are proportional to the real parameter $t \geq 0$.

Definition 2. *The number $t_* \geq 0$ is named the limiting parameter of loading and $(t_* f, t_* F)$ is named the limiting static load, if $(tf, tF) \in B$ for $0 \leq t \leq t_*$ and $(tf, tF) \notin B$ for $t > t_*$.*

For the ideal saturating material, the main problem is the investigation of the set of positive parameters t, for which the one-parametric functional

$$I_t(u) = \int_\Omega \Phi(x, \nabla u(x) + I)\, dx - tA(u)$$

is bounded from below on the set (2).

In practice the estimation from below for the limiting static load is interesting because this information is sufficient for estimation of the strength of non-linear elastic solids. As a result, the following basic result was proved by the author in [7, 8].

Theorem 1. *For the limiting parameter of loading the following estimation from below is true:*

$$t_* \geq t_- = \inf \left\{ \int_\Omega |\nabla u(x)| \varphi(x)\, dx : \quad u \in V, \quad A(u) = 1 \right\}, \tag{3}$$

$$\varphi(x) = \sup \left\{ \frac{\Phi(x, Q)}{|Q - I|} : \quad Q \in \mathbb{M}^3 \right\}.$$

According to the sense, $\varphi(x)$ is *the function of saturation*, it being known that $\varphi(x) > 0$ and for a homogeneous solid $\varphi = \text{const}(x)$.

From the Definition 2 it follows that for $t_- < 1$ the elastostatic variational problem (2) can have no solution. From the physical point of view this effect is treated as the global destruction of a solid. Therefore, the main problem for the estimation of the mechanical durability of non-linear elastic solids is *the limit analysis problem* (3).

3. Dual LAP

We construct here the dual LAP for a homogeneous non-linear elastic solid with $\varphi = \text{const}(x)$ by methods of the duality theory [16]. For this reason we rewrite the initial minimization problem (3) as the minimax variational problem using the relation $|Q| = \sup\{Q \cdot\cdot S : \quad |S| \leq 1\}$ which is true for every tensor $Q, S \in \mathbb{M}^3$, where $Q \cdot\cdot S = Q_i^\alpha S_i^\alpha$ is the double scalar product of tensors. As a result, we have the following problem:

$$t_- = \varphi \inf \{ \sup (L(u, S) : \quad S \in V^*) : \quad u \in V, \quad A(u) = 1 \},$$

where

$$L(u, S) = \int_\Omega \nabla u(x) \cdot\cdot S(x)\, dx, \qquad V^* = \{ S \in W^{1,\infty}(\Omega, \mathbb{M}^3) : \quad |S| \leq 1 \}.$$

For the bilinear functional $L(u, S)$ the classical equality of the duality theory

$$\inf_u \sup_S L(u, S) = \sup_S \inf_u L(u, S)$$

is true [16], as a result

$$t_- = \varphi \sup\{\, K(S): \quad S \in V^* \}\,. \tag{4}$$

Here $K(S)$ is the dual functional which is easily calculated by the method of Lagrange's multipliers and integration, taking into account the boundary condition on Γ^1:

$$K(S) = \inf\{\, L(u,S): \quad u \in V, \quad A(u) = 1\}$$
$$= \nu + \inf\left\{ \int_{\Gamma^2} (n{\cdot}S - \nu F){\cdot}u\, d\gamma - \int_{\Omega} (\nabla{\cdot}S + \nu f){\cdot}u\, dx: \quad u \in V \right\},$$

where n is the unit normal vector for boundary Γ^2 and the number $\nu > 0$ is the Lagrange's multiplier for the condition $A(u) = 1$.

The dual functional is proper on V^* (i.e. $K \not\equiv -\infty$ [16]) only if $S = \nu\Sigma$, where Σ belongs to *the set of admissible non-symmetrical Piola–Kirchhoff stress tensors*

$$G = \{\Sigma \in W^{1,\infty}(\Omega, \mathbb{M}^3): \quad \nabla{\cdot}\Sigma + f = 0 \quad \text{in} \quad \Omega, \quad n{\cdot}\Sigma = F \quad \text{on} \quad \Gamma^2\}. \tag{5}$$

Then the problem (4) has the form $t_- = \varphi \sup\{\, \nu > 0: \quad \nu \le |\Sigma|^{-1}, \quad \Sigma \in G\}$.

For admissible tensors the norm in the space of continuous functions can be correctly defined as $|||\Sigma||_0 = \max\{|\Sigma(x)|: \quad x \in \Omega\} < \infty$ [15]. Therefore, the relation $t_- = \varphi/\tau$ is true, where the parameter τ is the solution of *the dual LAP*

$$\tau = \inf\{\, \|\Sigma\|_0: \quad \Sigma \in G\}\,. \tag{6}$$

According to the sense, τ is *the dual limiting parameter of loading.*

As a result, the estimation of the mechanical durability of a homogeneous non-linear elastic solid reduces to finding an admissible non-symmetrical Piola–Kirchhoff stress tensor of minimal intensiveness which equilibrates with the external forces. This problem is fully correct because the admissible tensor has *nine* independent components satisfying only *three* differential equations, i.e. a minimization in *six* independent components is possible.

4. Finite-element approximation of dual LAP

By the standard FEA for the domain $\Omega \subset \mathbb{R}^n$ $(n = 1,2,3)$ the sets $\Omega_h = \cup T_h$ and $\Gamma_h = \partial\Omega_h$ are constructed such that $|\Omega\backslash\Omega_h| \to 0$ and $|\Gamma\backslash\Gamma_h| \to 0$ for $h \to +0$ regularity, where h is the characteristic step of approximation and T_h is the simplest simplex [17]. Here the symbol $|U|$ denotes Lebesgue's measure of the appropriate open set U. Every FEA is described by the set of nodes $\{x^k\}_{k=1}^m$.

For the admissible tensors the piecewise linear continuous approximation is used [17]:

$$\Sigma_h(x) = S^k \Psi_k(x) \qquad (k = 1,2,\ldots,m),$$

where $S^k \in \mathbb{M}^3$ is the admissible tensor at the node x^k, $\Psi_k: \Omega_h \to \mathbb{R}$ is the continuous and linear simplex scalar function on every simplex such that $\Psi_k(x^r) = \delta_{kr}$ $(k,r = 1,2,\ldots,m)$. Here δ_{kr} is the Kronecker symbol. The supp (Ψ_k) consists of simplexes having the node x^k as common.

The set of admissible tensors is approximated by the set

$$G_h = \Big\{ S^k \in \mathbb{M}^3: \quad S^k{\cdot}\nabla\Psi_k(x) + f_h(x) = 0, \quad x \in \forall T_h \subset \Omega_h;$$
$$n_h(x^r){\cdot}S^k\Psi_k(x^r) = F_h(x^r), \quad x^r \in \Gamma_h^2 \Big\}, \tag{7}$$

which is the *convex set with linear boundaries* in the space of global variables $\mathbb{R}^{9m}$. The equation (7) uses the standard piecewise linear continuous FEA of external forces (f_h, F_h) and normal vector n_h for the boundary Γ^2.

As a result, the dual LAP (6) is approximated by the problem of mathematical programming with linear limitations as equalities

$$\tau_h = \min \left\{ \max \left(|S^k| : \quad k = 1, 2, \ldots, m \right) : \quad S^k \in G_h \right\}. \tag{8}$$

If the number of finite elements equals m_1 and the number of nodes on Γ_h^2 equals m_2, then the number of free variables in the problem equals $9m - (3m_1 + m_2)$. It is easily seen that the minimal number of variables equals $8n + 5\,(n = 1, 2, 3)$ which is reached for the domain coinciding with the simplest n-dimension simplex because in this case $m = n + 1$, $m_1 = 1$, $m_2 = n + 1$.

The objective function in the problem (8) is the linear combination of convex hypercones in the space $\mathbb{R}^{9m}$. Therefore, due to linearity of limitations in the set G_h this finite dimensional problem is effectively solved by the standard *method of gradient projection* which is easily adapted for parallel calculations.

5. Numerical results

In the numerical experiments the following BVP was considered: a finite round rod is stretched in a test machine of the rigid type by a given axial force P, and the problem is assumed to be an axially symmetric one. In this case, the map is described by the following relation using non-dimensional cylindrical co-ordinates:

$$X(\rho, \varphi, z) = x\,(a\rho + a\,r(\rho, z), \varphi, lz + lw(\rho, z)),$$

$$C(r, w) = \nabla u = \nabla X - I = \begin{pmatrix} \partial r/\partial \rho & 0 & \eta^{-1}\,\partial w/\partial \rho \\ 0 & r/\rho & 0 \\ \eta\partial r/\partial z & 0 & \partial w/\partial z \end{pmatrix},$$

where $\rho \in [0, 1]$, $\varphi \in [0, 2\pi)$, $z \in [0, 1]$, $\eta = a/l$, a and l are the radius of the section and the semilength of the rod, respectively, r and w are the radial and axial displacements.

The material of the rod is assumed to be homogeneous and compressible. It is described by the elastic potential

$$\Phi(Q) = \sqrt{3}\mu\,(|Q| - |I|) + \frac{1}{2}k_0\,(\det(Q) - 1)^2,$$

where $\mu > 0$ and $k_0 > 0$ is the shear and bulk modulus, respectively [1–9].

For the limiting stretching force P_* the estimation from below $P_* \geq \sqrt{3}\mu\pi a^2 t_-$ is true, where the parameter t is the solution of the initial LAP (3) which, in view of the axial symmetry, has the following form:

$$t_- = \inf \left\{ 2\int_0^1 \int_0^1 |C(r, w)|\rho\, d\rho dz : \quad (r, w) \in V, \quad w(\rho, 1) = 1 \right\}, \tag{9}$$

$$V = \left\{ (r, w) \in \left(W^{1,1}((0, 1) \times (0, 1))\right)^2 : r(0, z) = 0, \frac{\partial r}{\partial z}(\rho, 0) = 0,\ w(\rho, 0) = 0,\ r(\rho, 1) = 0 \right\}.$$

It is easily verified that the parameter sought for $t_- = \sqrt{3}/(2\tau)$, where the parameter τ is a solution of the appropriate dual LAP (6) on the following set of admissible non-symmetrical Piola–Kirchhoff stress tensors:

$$G = \left\{ \Sigma \in W^{1,\infty}(\Omega, \mathbb{M}^3) : \quad \nabla \cdot \Sigma = 0 \text{ in } \Omega, \quad \Sigma_{zz} = 1 \text{ on } \Gamma^2 \right\}. \tag{10}$$

Here $\Omega = (0, 1) \times (0, 1)$ and $\Gamma^2 = \{\rho \in [0, 1], z = 1\}$.

Table 1. The dual limiting parameter of loading for the uniform $N \times N$ triangulations of the domain

N	5	10	20	40	80
τ_h	2.56	2.02	1.76	1.04	1.01

In the computer experiments a uniform $N \times N$ triangulation of the domain Ω was used. As a result, the problem of mathematical programming (8) was solved for $9(N+1)^2$ variables satisfying $6N^2 + 4N$ linear limitations as equalities. In the following table the experimental results are shown. It is easily seen that $\tau_h \searrow 1$ with increase of N.

The above analytical and numerical results are new. They have a practical interest and need more theoretical and experimental research.

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