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Numerical simulation of random fields using correlated random vector and the Karhunen-Loève expansion

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This paper presents an approach of one- and two-dimensional random field simulation methods using a correlated random vector and the Karhunen-Loève expansion. Comparison of the authors' analytical solution of the Fredholm integral equation of the second kind with the numerical solution using the finite element method and the inverse vector iteration technique is presented. Numerical approach and sample realizations of one- and two-dimensional random fields are presented using described techniques as well as generated probability distribution functions for chosen point of the analysed domain.

Keywords: eigenproblem, Karhunen-Loève expansion, random field, Lalescu-Picard equation.

1. Introduction

In the structure of real materials, the values of physical and mechanical parameters and thus the state functions can change randomly between points. These values of parameters are determined based on the results of experimental studies with the support of statistical analysis. In the case of physical properties of engineering materials used in practice, this analysis is usually carried out with one random variable. On the basis of the statistical population obtained from sampling at specified points, the central trend (expected value) and the measure of dispersion (usually the standard deviation) are determined. This description may not be sufficient, because these characteristics also change in space. The stochastic process $\alpha(t,\omega)$ being a continuous or discrete function assigning to the time argument t a random variable $\alpha(\omega)$ can be used as a more appropriate model. Random fields can be used to describe real nature of parameters in geotechnics [7] (soil analysis), civil engineering and material engineering [8] (concrete, wood, composites analysis), mechanical engineering (machinery parts analysis) and many others [5, 11, 13].

A random field is a function of an infinite number of random variables [10]. On the random fields there are performed some mathematical operations, aimed at obtaining the solution of the problem under consideration. Some operations, however, lead to significant numerical difficulties, often making it impossible to get a solution. To obtain a numerically useful form of the process $\alpha(x,\omega)$, the so-called canonical expansion is used, consisting in presenting the process as a sum of the expected value and cross-uncorrelated random variables. One of the many forms of such representation found in the literature is the Karhunen-Loève expansion (KL). Another common

approach which can be used is a correlated random vector (CRV) [3]. In this method, a continuous covariance function is presented in the matrix form of the desired dimension. Then, the resulting vector is obtained as a product of a random uncorrelated vector and a lower triangular matrix obtained from the Cholesky decomposition of the covariance matrix. This paper presents a coherent description of the analytical solution of the Fredholm integral equation of the second kind and presents a numerical solution using the finite element method and the inverse vector iteration technique. Sample realizations of one- and two-dimensional random fields are presented using described techniques as well as generated probability distribution functions for the chosen point. It is also shown that four-dimensional covariance matrix used to simulate a two-dimensional random field that can be interpreted as a two-dimensional block matrix, which allows using the one-dimensional approach straightforward.

2. The karhunen-loève expansion

Generalization of the stochastic process notion in case of a random, multidimensional function of coordinates is a random field [9], which can be defined as follow:

A random field $\alpha(x,\omega)$ [9] where $(x \in \mathsf{D} \subset \mathbb{R},\ \omega \in \Omega)$ we call a function, in which each point x assigns a random variable $\alpha(\omega)$ defined in a fixed probability space $(\Omega, \mathsf{Z}, \mathsf{P})$. Here, Ω is the set of elementary events, Z is the σ-algebra and $\mathsf{P} : \mathsf{Z} \to [0,1]$ is a probability measure.

In order to describe a random field, the expected value is most commonly used:

$$E\left(\alpha(x,\omega)\right) = \int_{\Omega} \alpha(x,\omega)\, d\mathsf{P} \tag{1}$$

and covariance function

$$C(x,t) = \int_{\Omega} \left(\alpha(x,\omega) - E\left(\alpha(x,\omega)\right)\right)\left(\alpha(t,\omega) - E\left(\alpha(t,\omega)\right)\right) d\mathsf{P}. \tag{2}$$

In this paper, the Gaussian probability space will be adopted, where $\Omega \in (-\infty, \infty)$.

A computationally useful representation of the process $\alpha(x,\omega)$ can be presented with a KL [4–6]:

$$\alpha(x,\omega) = \xi_0 E\left(\alpha(x,\omega)\right) + \sum_{i=1}^{\infty} \xi_i(\omega)\sqrt{\lambda_i} f_i(x). \tag{3}$$

This expansion is optimal in the Fourier sense because it minimizes the mean square error resulting from truncating to the first M terms:

$$\alpha(x,\omega) \approx \xi_0 E\left(\alpha(x)\right) + \sum_{i=1}^{M} \xi_i(\omega)\sqrt{\lambda_i} f_i(x). \tag{4}$$

In Eqs (3) and (4) $\xi(\omega)$ is a Gaussian random variable with zero mean and standard deviation equal to one ($\xi_0 = 1$). Constants λ and functions $f(x)$ are respectively the eigenvalues and eigenfunctions, obtained from the solution of the Fredholm second kind integral equation with the covariance function as the kernel [4, 12]:

$$\int_{\mathsf{D}} C(x_1,x_2) f_i(x_2) dx_2 = \lambda_i f_i(x_1). \tag{5}$$

To solve Eq. (5), it is necessary to define the covariance function $C(x_1,x_2)$. Obtaining a solution of this equation in a closed form is possible only for selected covariance functions, which can be found in [7]. One of the most common functions, among others, used in [2, 4, 6] is the function (Fig. 1):

$$C(x_1,x_2) = \sigma_\alpha^2 \cdot e^{\frac{-|x_1-x_2|}{b}}, \tag{6}$$

where σ_α^2 is the coefficient of variance, and b is a correlation length.

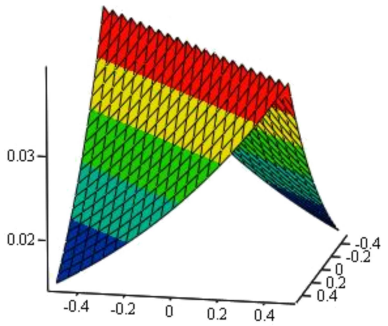

Fig. 1. Covariance function plot according to Eq. (6). $\sigma_\alpha = 1$.

3. Correlated random vector approach

Suppose that the stochastic process is approximated by the vector of random variables, namely

$$\alpha(x,\omega) \xrightarrow{\text{approx.}} \boldsymbol{\alpha}(\omega) \qquad x \in \langle -a, a\rangle . \tag{7}$$

Therefore, the covariance function $C(x_1, x_2)$, $x_1, x_2 \in \langle -a, a\rangle$ takes the form of the following matrix:

$$\mathbf{C} = C_{i,j} = C\left(-a + i\Delta x_1, -a + j\Delta x_2\right), \tag{8}$$

where

$$\Delta x_1 = \Delta x_2 = \frac{2a}{p}, \tag{9}$$

where p is the number of division. The random vector is described by the relation

$$\boldsymbol{\alpha}_{\text{vec}} = \mathbf{LN}, \tag{10}$$

where $\mathbf{N}_{(px1)}$, is a vector, of which elements are random variables with a normal distribution with $\mu = 0$ and $\sigma = 1$, while $\mathbf{L}_{(pxp)} = Chol(\mathbf{C}_{(pxp)})$ is a lower triangular Cholesky matrix determined by the formula [3]:

$$\mathbf{C} = \mathbf{L}\mathbf{L}^T. \tag{11}$$

A similar procedure can be performed for a two-dimensional field

$$\alpha(x,y,\omega) \xrightarrow{\text{approx.}} \boldsymbol{\alpha}(\omega), \qquad x \in \langle -a, a\rangle, \quad y \in \langle -a, a\rangle, \tag{12}$$

where the four-dimensional covariance function

$$C(x_1, x_2, y_1, y_2), \qquad x_1, x_2, y_1, y_2 \in \langle -a, a\rangle \tag{13}$$

can be presented in the form

$$\mathbf{C} = C_{i+k(p+1),j+m(p+1)} = C(-a + m\Delta x_1, -a + k\Delta x_2, -a + j\Delta y_1, -a + i\Delta y_2) \tag{14}$$

or in the matrix notation

$$\mathbf{C} = \begin{bmatrix} \widehat{\mathbf{C}}_{0,0} & \cdots & \widehat{\mathbf{C}}_{0,m} \\ \vdots & \ddots & \vdots \\ \widehat{\mathbf{C}}_{k,0} & \cdots & \widehat{\mathbf{C}}_{k,m} \end{bmatrix}, \tag{15}$$

where

$$\Delta x_1 = \Delta x_2 = \Delta y_1 = \Delta y_2 = \frac{2a}{p}. \tag{16}$$

In this case, the random vector is described by the relation

$$\boldsymbol{\alpha} = \alpha_{i,j} = \left(\alpha_{\text{vec}}\right)_{j+i(p+1)}, \tag{17}$$

where Eq. (10) is used, but the vector $\mathbf{N}_{(p^2x1)}$ and matrix $\mathbf{L}_{(p^2xp^2)} = Chol(\mathbf{C}_{(p^2xp^2)})$ have different dimensions.

4. Analytical and numerical solution of the fredholm second kind equation

In order to determine eigenvalues and eigenvalues for a given covariance function, a solution of the Fredholm second kind integral Eq. (5) should be found. This equation can be transformed into the Lalescu-Picard equation, i.e.,

$$\lambda f(x) = \sigma^2 \left(e^{-cx} \int_{-a}^{x} e^{ct} f(t) dt + e^{cx} \int_{x}^{a} e^{-ct} f(t) dt \right), \qquad x \in \langle -a, a \rangle \quad c = \frac{1}{b}. \tag{18}$$

By substituting

$$G(x) = \int_{-a}^{x} e^{ct} f(t) dt, \qquad H(x) = \int_{x}^{a} e^{-ct} f(t) dt \qquad \text{and} \qquad g(x) = e^{cx}, \qquad h(x) = e^{-cx},$$

Eq. (18) can be presented in a concise form:

$$\lambda f(x) = \sigma^2 \left(h(x) G(x) + g(x) H(x) \right). \tag{19}$$

Using the derivative of Eq. (19) relative to x:

$$\lambda f'(x) = \sigma^2 \left(h'(x) G(x) + g'(x) H(x) \right) \tag{20}$$

and Eq. (19), a system of equations can be built, which must be solved for $H(x)$ and $G(x)$. After substituting the obtained relations to the second derivative of Eq. (19) with respect to x, the second-order differential equation, written in the matrix form, is obtained:

$$\lambda \left\| \begin{bmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{bmatrix} \right\| - \sigma^2 \left(\left\| \begin{bmatrix} g(x) & h(x) \\ g'(x) & h'(x) \end{bmatrix} \right\| \right)^2 f(x) = 0. \tag{21}$$

After appropriate transformations Eq. (21) takes the form:

$$f''(x) + \omega^2 f(x) = 0, \tag{22}$$

where

$$\omega^2 = \frac{2c\sigma^2}{\lambda} - c^2. \tag{23}$$

The general solution of Eq. (22) has the form:

$$f(x) = a_1 \cos(\omega x) + a_2 \sin(\omega x). \tag{24}$$

In order to determine the parameters a_1 and a_2 Eqs (22) and (24) as well as condition $-a \le x \le a$ should be used. From the above, a system of equations is obtained:

$$\begin{bmatrix} c - \omega \tan(\omega a) & \omega + c \tan(\omega a) \\ c - \omega \tan(\omega a) & -(\omega + c \tan(\omega a)) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{25}$$

Non-trivial solutions for the above homogeneous system exist if the determinant of the coefficient matrix is equal to zero, e.g.,

$$\left\| \begin{bmatrix} c - \omega \tan(\omega a) & \omega + c \tan(\omega a) \\ c - \omega \tan(\omega a) & -(\omega + c \tan(\omega a)) \end{bmatrix} \right\| = 0. \tag{26}$$

As a result, two transcendental equations are obtained whose elements are the searched values ω. Using the transformed Eq. (23)

$$\lambda_i = \frac{2c\sigma^2}{\omega_i^2 + c^2}, \tag{27}$$

eigenvalues λ can be determined. By normalizing the system of equations (25) with respect to the parameters a_1 and a_2 the i-th solution of the specific differential Eq. (22) can be obtained for i-th eigenvalue, e.g.,

$$f_i(x) = \begin{cases} \dfrac{\omega_i \cos(a\omega_i - \omega_i x) + c\sin(a\omega_i - \omega_i x)}{\omega_i \cos(a\omega_i) + c\sin(a\omega_i)} & \text{for even } i, \\ \dfrac{c\sin(a\omega_i + \omega_i x) - \omega_i \cos(a\omega_i + \omega_i x)}{\omega_i \sin(a\omega_i) - c\cos(a\omega_i)} & \text{for odd } i. \end{cases} \tag{28}$$

For a two-dimensional random field, the Fredholm second kind equation takes the form:

$$\int_{\mathsf{D}} C(x_1, x_2, y_1, y_2) f_n(x, y) dy = \lambda_n f_n(x, y). \tag{29}$$

The covariance kernel can be represented as the product of two functions

$$C(x_1, x_2, y_1, y_2) = C_x(x_1, x_2) C_y(y_1, y_2), \tag{30}$$

where

$$C_x(x_1, x_2) = \sigma_x \cdot e^{-\frac{|x_1 - x_2|}{b_x}} \tag{31}$$

and

$$C_y(y_1, y_2) = \sigma_y \cdot e^{-\frac{|y_1 - y_2|}{b_y}}. \tag{32}$$

In Eqs (31) and (32), σ_x i σ_y are the square roots of the coefficients of variance, and b_x and b_y are the correlation lengths. Assuming that the eigenfunctions can be represented in the form

$$f_n(x, y) = f_i(x) f_j(y) \tag{33}$$

and eigenvalues in the form

$$\lambda_n = \lambda_i \lambda_j, \tag{34}$$

the KL for a two-dimensional random field takes the form:

$$\alpha(x, y, \omega) = \xi_0 E\left(\alpha(x, y)\right) + \sum_{j=1}^{M} \sum_{i=1}^{M} \xi_{i,j}(\omega) \sqrt{\lambda_i \lambda_j} f_i(x) f_j(y). \tag{35}$$

In Eq. (35) eigenvalues and eigenfunctions from the one-dimensional solution of the Eq. (5) can be used.

The basis for the numerical solution using the finite element method is the discretisation of eigenfunctions:

$$f_k(x) = \mathbf{f}_k^T \mathbf{h}(x), \tag{36}$$

where $\mathbf{f}_k$ is a vector of coefficients, and $\mathbf{h}(x) = [N_1^e(x), N_2^e(x), N_3^e(x)]^T$ is a vector of shape functions. The following quadratic shape functions were adopted in this work:

$$N_1^e(x) = \frac{(x - x_j)(x - x_k)}{(x_i - x_j)(x_i - x_k)},$$

$$N_2^e(x) = \frac{(x - x_i)(x - x_k)}{(x_j - x_i)(x_j - x_k)},$$

$$N_3^e(x) = \frac{(x - x_i)(x - x_j)}{(x_k - x_i)(x_k - x_j)}.$$

By substituting function (36) into the integral Eq. (5) and taking into account that $\boldsymbol{\lambda} = \delta_{i,j}\lambda_i$ it can be written:

$$\mathbf{Kf} = \boldsymbol{\lambda}\mathbf{Mf}, \tag{37}$$

where

$$\mathbf{K} = K_{3i+n,3j+m} = \int_{(-a+j\Delta x_2)}^{(-a+(j+1)\Delta x_2)} \int_{(-a+j\Delta x_1)}^{(-a+(j+1)\Delta x_1)} C(x_1, x_2)\, h_n(x_2)\, h_m(x_1)\, dx_1 dx_2, \tag{38}$$

$$\Delta x_1 = \Delta x_2 = \frac{2a}{d},$$

$$\mathbf{M} = M_{3i+n,3j+m} = \int_{(-a+j\Delta x_1)}^{(-a+(j+1)\Delta x_1)} h_n(x_1)\, h_m(x_1)\, dx_1, \tag{39}$$

$$\Delta x_1 = \frac{2a}{d}.$$

In this work, the inverse vector iteration method was used to solve the eigenproblem (37), the description of which can be found in a large number of publications [1]. A comparison of eigenvalues and eigenvalues obtained from the analytical and numerical methods, for two values of the correlation length, $b = 0.1$ and $b = 1.0$ (Figs 2–6), is shown below. The analysed domain was divided

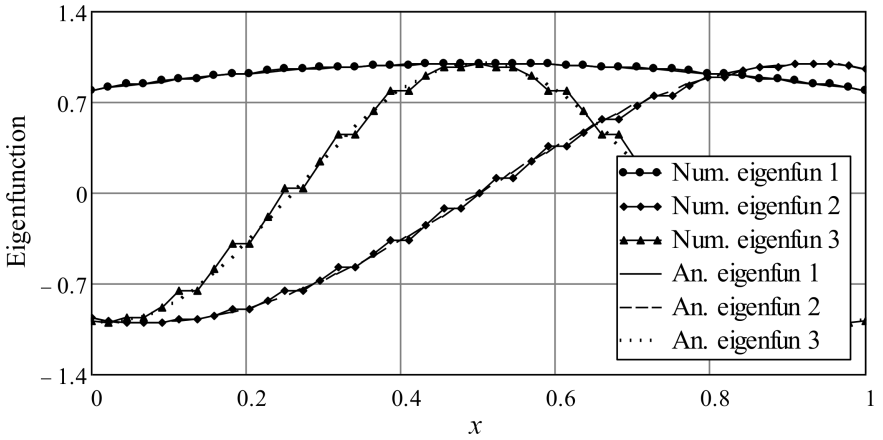

Fig. 2. Eigenfunctions 1 to 3 according to Eqs (28) and (36). Correlation length $b = 1.0$.

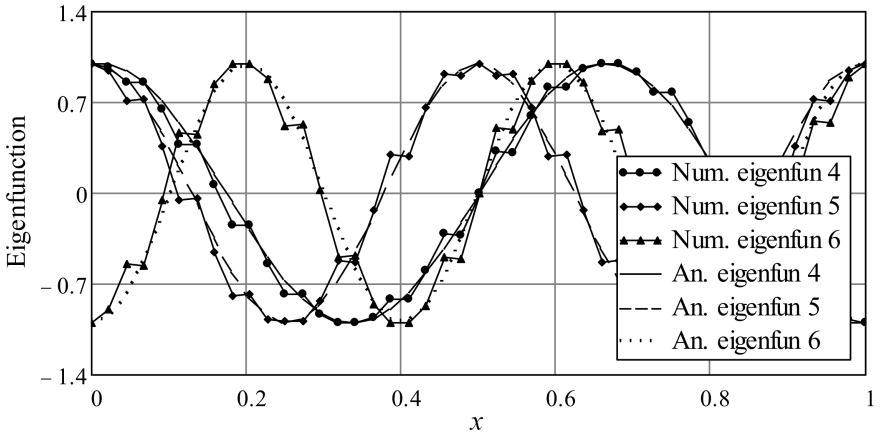

Fig. 3. Eigenfunctions 4 to 6 according to Eqs (28) and (36). Correlation length $b = 1.0$.

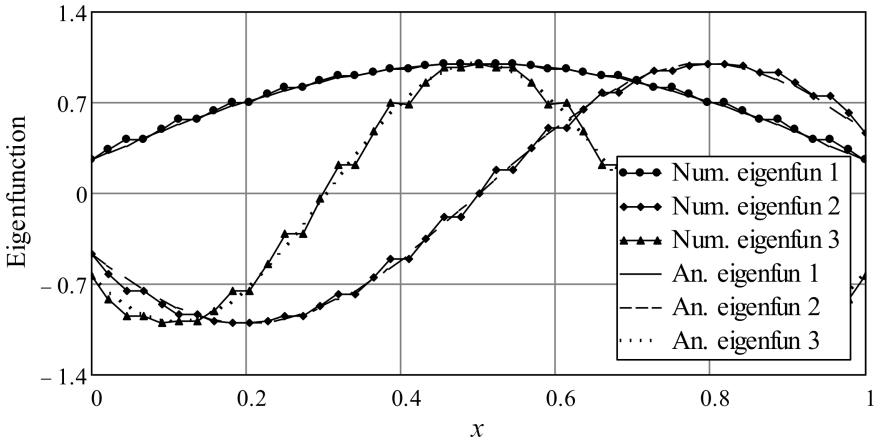

Fig. 4. Eigenfunctions 1 to 3 according to Eqs (28) and (36). Correlation length $b = 0.1$.

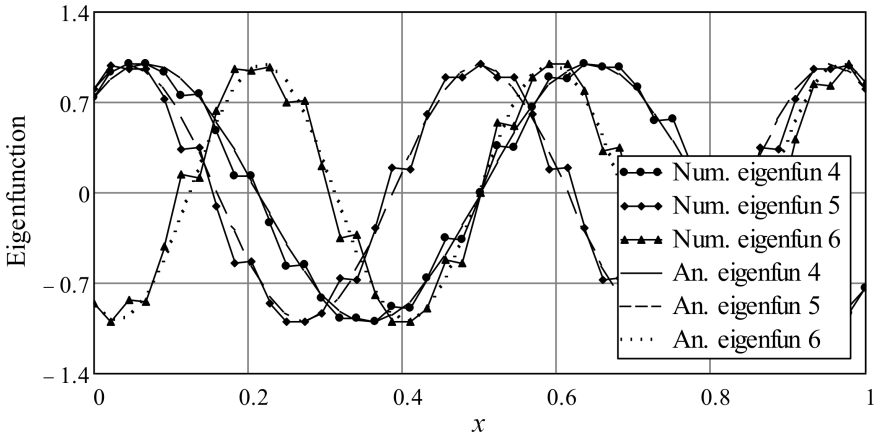

Fig. 5. Eigenfunctions 4 to 6 according to Eqs (28) and (36). Correlation length $b = 0.1$.

into 44 finite elements ($p = 44$) of equal length. The analysis shows that near the functions roots (Figs 2–5) the numerical solution becomes unstable.

The instability increases with the degree of eigenfunctions, and with a reduction in the number of finite elements. In the case of the eigenvalue, a very good agreement of results was found.

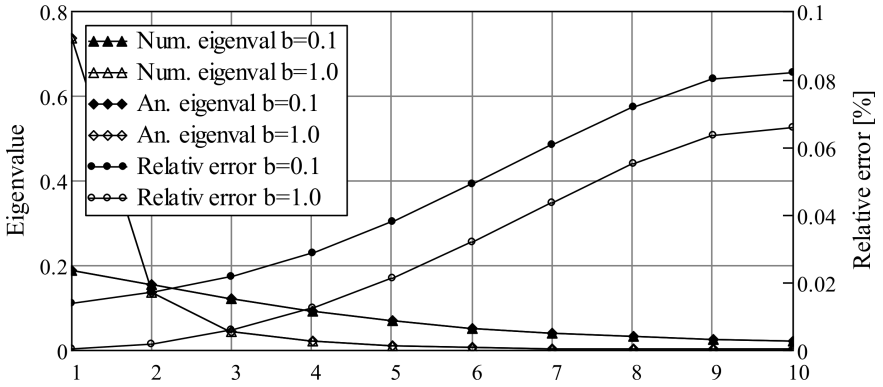

Fig. 6. Eigenvalues of the process from the numerical and analytical solution.

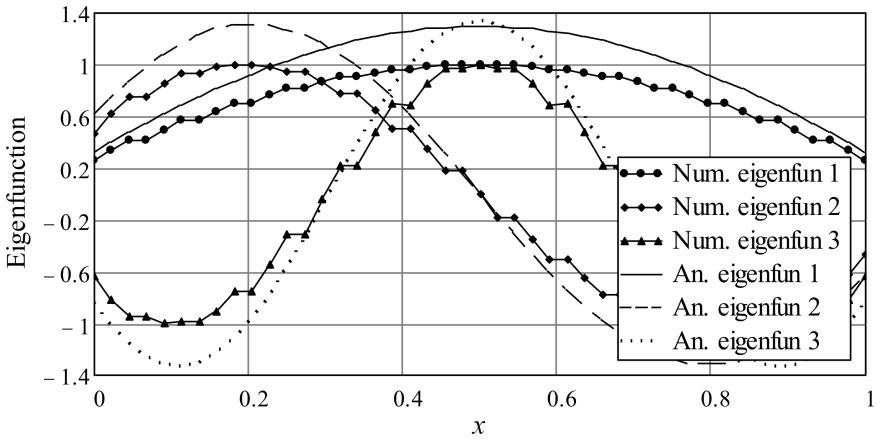

Fig. 7. Eigenfunctions 1 to 3 according to Eq. (36) and solution of R.G. Ghanem and P.D. Spanos [4]. Correlation length $b = 0.1$.

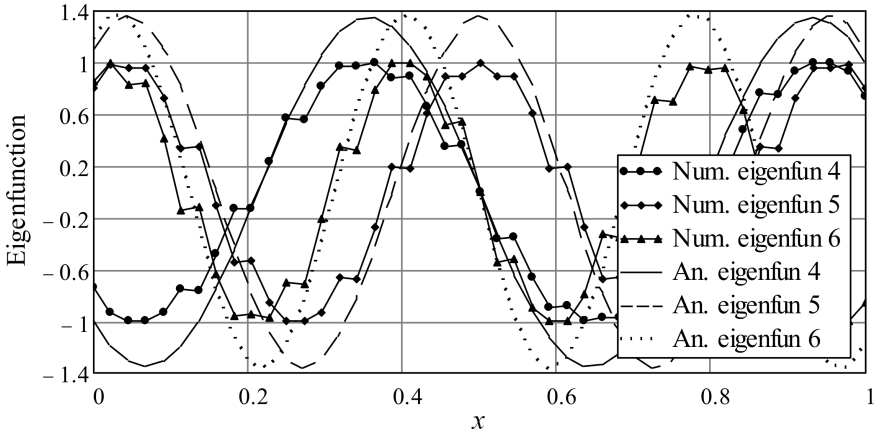

Fig. 8. Eigenfunctions 4 to 6 according to Eq. (36) and solution of R.G. Ghanem and P.D. Spanos [4]. Correlation length $b = 0.1$.

The relative error of the numerical and analytic solution of this value is shown in Fig. 6. In research related to the stochastic analysis of mechanics problems, the solution given in the work of R.G. Ghanem and P.D. Spanos [4] is often used. A comparison of the numerical solution with the analytical solution from [4] is presented in Figs 7 and 8.

5. Numerical simulation of one- and two-dimensional random fields

In this section, the results of random field simulations are shown. To perform these simulations, the crude Monte Carlo method was used. In this method, in each realization vector is sampled using Eq. (10). The solution was obtained based on a specific number of realizations $n = 5000$ of the vector $\mathbf{N}$. For computation, an appropriate matrix $\mathbf{C}$ should be determined. Contour plots of this matrix for one- and two-dimensional case are shown below with the coefficient of variance equal to one (Figs 9 and 10).

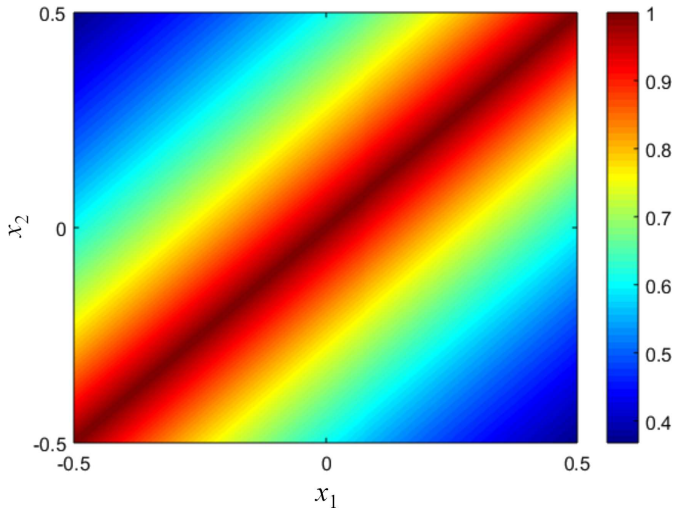

Fig. 9. Contour plot of function (6).

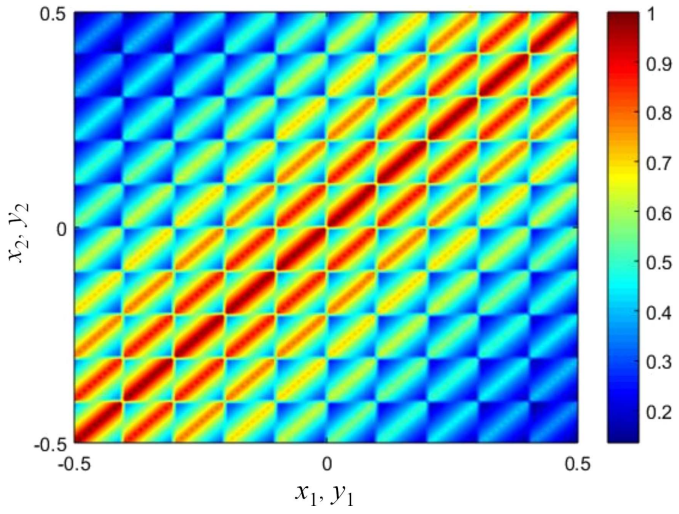

Fig. 10. Contour plot of function (30).

Example realizations of one- and two-dimensional random fields using the analytical and numerical approach are presented below (Figs 11–13).

To verify the obtained random fields, data from one chosen point was used to build probability distribution. Both analytical and numerical approach give appropriate results which were compared with normal distribution function (mean and standard deviation was calculated from the obtained data). For samples near the expected value, there is a bigger dispersion in both methods. For the KL with decreasing number of expansion terms M mean value becomes larger and standard deviation lower than values obtained with the CRV method.

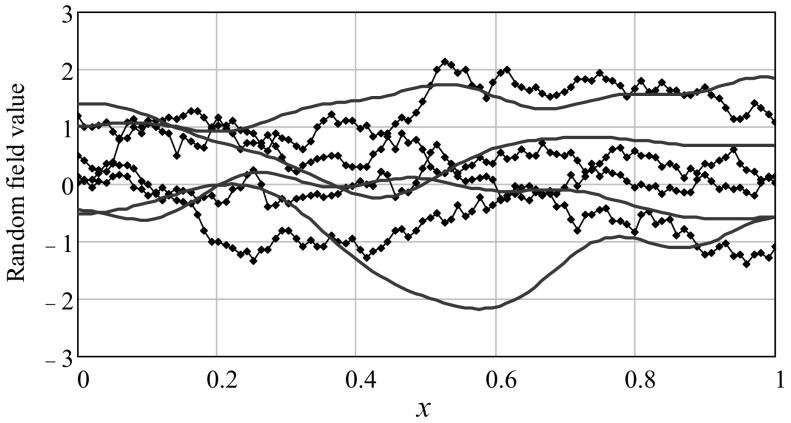

Fig. 11. Example realization of one-dimensional random field using the analytical approach (continuous line) and the numerical approach (dotted line).

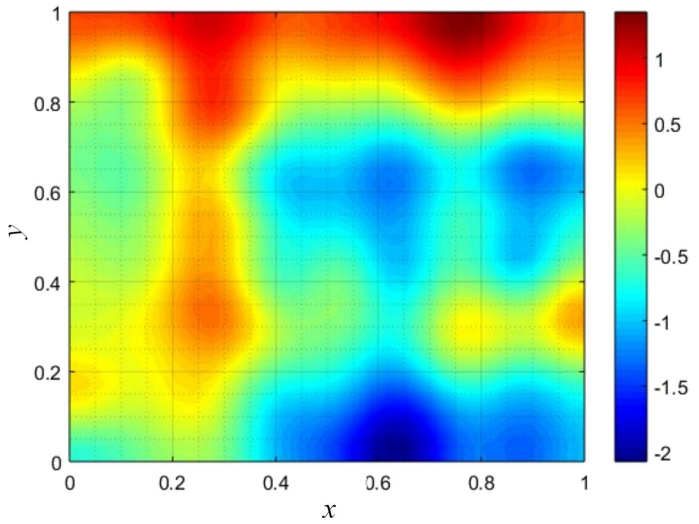

Fig. 12. Example realization of two-dimensional random field using the analytical approach.

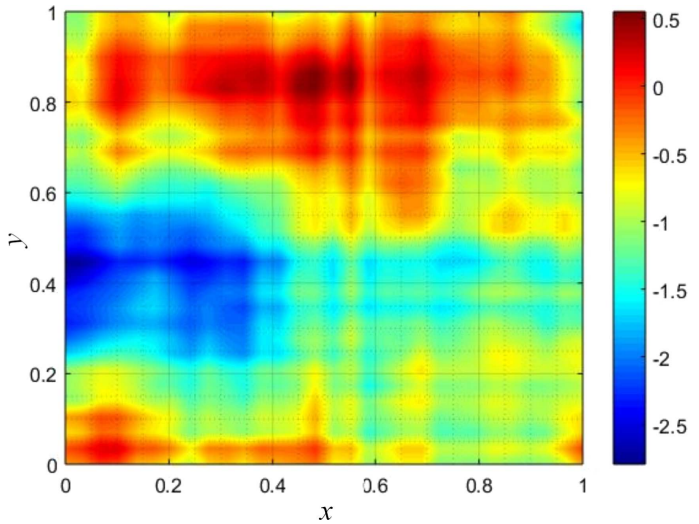

Fig. 13. Example realization of two-dimensional random field using the numerical approach.

6. Conclusions

This paper presents the analytical and numerical solution of the Fredholm second kind integral equation. The analytical solution has been performed using the transformation to the Lalescu-Picard equation, and the numerical solution using the finite element method and the inverse vector iteration technique. The analytical solution is in very good agreement with the numerical solution, in contrast to the analytical solution given by R.G. Ghanem and P.D. Spanos [4] (Figs 7 and 8). The analysis shows that near the eigenfunctions roots (Figs 2–5) the numerical solution becomes unstable. This instability increases with the eigenfunctions degree and with decreasing number of finite elements. This may be related to the occurrence of singular points in the covariance function for correlated random variables. Similarly, for the probability distributions of both methods near the expected value, there is a larger dispersion of samples (Figs 14 and 15).

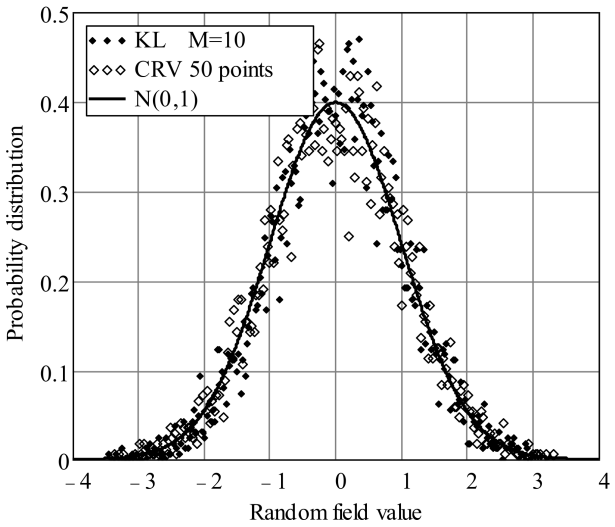

Fig. 14. Probability distribution obtained from the Monte Carlo simulations for a chosen point (one-dimensional case) using the CRV and the KL. Comparison with the normal distribution function.

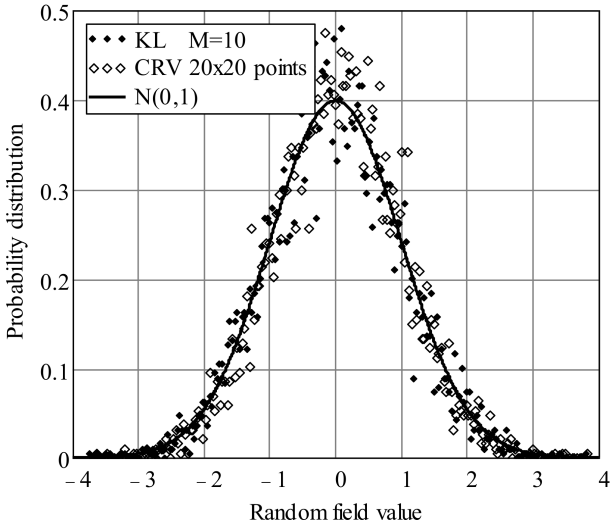

Fig. 15. Probability distribution obtained from the Monte Carlo simulations for a chosen point (two-dimensional case) using the CRV and the KL. Comparison with the normal distribution function.

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