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Real ant colony optimization as a tool for multi-criteria problems

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This paper presents a population-based heuristic method – a real ant colony optimization (RACO) as a tool for multi-criteria optimization problems. The idea of multi-criteria optimization is discussed and the necessary modifications of RACO are proposed. These modifications made possible to use the method to simultaneously search many Pareto-optimal solutions. The method was numerically tested in problems of benchmark-type and used for solving simple engineering problems. This article presents and discusses all results obtained in tests, and two different approaches to multi-criteria optimization are additionally compared (search then decision and decision then search).

Keywords: multi-objective optimization.

1. Introduction

In many engineering problems, an optimization of parameters (decision variables) that in the best way simultaneously fulfil several objectives is increasingly expected. We speak then about the so-called multi-criteria optimization, in which the solution is a set of parameters describing a compromise between various objectives [3]. The solution of such a problem, by its nature, is not unique because the optimization obtains (infinitely) many non-dominated solutions, generally known as the Pareto-optimal solutions [9], which are searched. A pair of solutions is non-dominated if one of them is better than the other with respect to at least one criterion and, at the same time worse with respect to another criterion. These types of solutions form a set whose individual solutions cannot be compared with each other. They are just possible variants of the solution of the optimization problem.

Having a set of solutions and knowing that each of them should be considered as optimal (in Pareto's sense) requires an additional assessment whose result could decide about the final selection from many proposed variants. Such an assessment may include the consideration of non-numerical criterion (e.g., the ease of implementation or production, aesthetics, etc.) made by a human decision maker. This additional evaluation is particularly important at the stage of design of technology devices or any other type of problem. Not without significance is also the information (about possible compromises) posed by the analysis of the points on the Pareto front.

An important aspect of multi-criteria optimization problems is to distinguish the process of searching and decision making [6]. In the first case, the optimization process is to lay the greatest possible number of Pareto-optimal solutions. In the second aspect, a human decision maker is necessary to make the often difficult trade-offs between conflicting objectives.

Depending on how the optimization and the decision process are combined, we get different ways of solving the multi-criteria problems. The approach understood as a *decision making before searching* means that the objectives of the multi-criteria optimization problems are aggregated into a single objective. In such a case, an aggregate function φ_s is optimized, whose arguments are the components of the vector of criteria, i.e., $f^s(\mathbf{x}) = \varphi_s(f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}))$. The choice of an aggregate function depends on the initial decision made by the decision maker and determines the result of the search process. One of the most frequently used methods is a weighted sum of criteria, in which the choice of weights determines which solution on the Pareto front will be found. Using the approach of decision making before searching, any optimization method can be used. It should be noted that this approach provides an opportunity to find only a single solution lying on the Pareto front [1, 11, 13]. In fact, in this approach the multi-objective problem is solved by translating it back to a single (or a series of) single objective, scalar problems.

In *search before decision making* the optimization is performed without any preference given to information. The result of the search process is a set of (ideally Pareto-optimal) candidate solutions from which the final choice is made by the decision maker. When choosing the approach which requires exploration of many solutions lying on the Pareto front, it is recommended to use population-based optimization methods. The appropriate modification of heuristics such as an evolutionary algorithm (EA) or other methods from the swarm intelligence group, allows to find many solutions simultaneously.

2. Pareto-optimal solution

As it was mentioned, in the multi-objective problem, the optimization can many possible compromises rather than a single solution. So, a multi-objective optimization does not produce a unique solution but a set of so-called Pareto solutions, the possible variants of the optimization problem solution. The obtained set can be called both the "trade-off surface" or the Pareto frontier. On the Pareto frontier, none of the points is "dominated". By definition we say that point $\mathbf{x}$ dominates $\mathbf{y}$ (or $\mathbf{y}$ is dominated by $\mathbf{x}$), what is noted by $\mathbf{x} \prec \mathbf{y}$ if

$$\forall\ k = 1, \ldots, m; \quad f_k(\mathbf{x}) \le f_k(\mathbf{y}) \ \wedge\ \exists\ k \in \{1, \ldots, m\} \ \ f_k(\mathbf{x}) < f_k(\mathbf{y}).$$

In other words, a solution is **non-dominated** if it is better than the other solutions with respect to at least one criterion and at the same time worse with respect to another. The primary goal of multi-objective optimization techniques, in search before decision variant, is to find a set of non-dominated points, i.e., the *Pareto set* (Fig. 1).

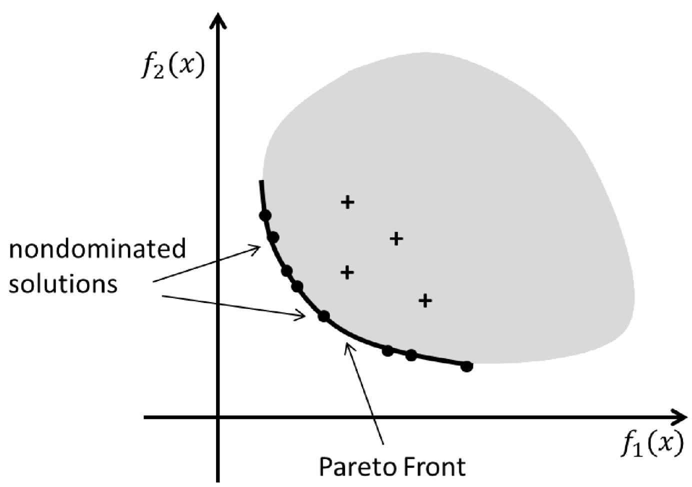

Fig. 1. The Pareto front.

3. Real ant colony optimization

In the present work, it was decided that multi-criteria problems are solved according to the search before decision scheme. This approach makes it possible to obtain, evaluate and compare many Pareto-optimal solutions, which may be important for optimization at the design stage of processes or devices. The real ant colony optimization [10] (RACO – one of the heuristic methods inspired by the behavior of ant colonies) was used to search the space of possible solutions. The method was modified to be able to solve multi-criteria optimization problems.

In general, the idea of the proposed algorithm is based on the formation of the trails so-called pheromone stains (possible solutions) of increasing intensity, which are produced by artificial ants. Intensity of the stains is associated with the solution quality which it represents. In the iteration procedure, new and better solutions are the results of ants' work, where each of them imposes a trail pheromone (creates a new proposition of solution). Although the ants' actions seem to be random, they are controlled by the use of appropriate probabilistic model. The construction of such a model is an essential part of RACO.

The modification of RACO, for the purposes of multi-criteria optimization, was to replace the standard selection by the ranking one, in which the solution rank depends on belonging to a given layer of non-dominated solutions. All solutions are divided into layers of mutually non-dominated solutions. The solution position on the ranking list is determined by the number of layers in which it is located.

In addition, the succession operation is used to keep the best stains (solutions) in the iteration and "evaporation" of the others. The process of creation of the new solutions was preserved in the form known from the original version of RACO [3, 10].

Generally, RACO consists of two parts:

1. **Initialization** in which the parameters of the algorithm are defined and initial archive of k pheromone "stains" (solutions) is generated; the "stains" are arranged with respect to their quality understood as belonging to the appropriate layer of non-dominated solutions;

2. **Computational loop** repeated until the stop criteria is not satisfied; in the single iteration, the new solutions are designed (by each of the ants), then the updated archive of solutions (pheromone "stains") is divided into the layers containing mutually non-dominated solutions.

The most important element of the presented heuristics is, of course, a *computational loop* where the new solutions are created as the result of work of each m ants. Each ant randomly selects a j-th solution ("stain" of pheromone) with the following probability:

$$p_j = \frac{\omega_j}{\sum\limits_{l=1}^{k} \omega_l},$$

where ω_j is the weight connected with j solution determined by using Gaussian distribution $g(\mu, \rho) = g(1, qk)$ (k – number of stains, q – parameter), *i.e.*,

$$\omega_j = \frac{1}{qk\sqrt{2\pi}} \cdot e^{\frac{-(j-1)^2}{2q^2k^2}}.$$

Weights reflect the pheromone "intensity" and the ants prefer to choose a seat/track that is more saturated with pheromones.

Next, the ants sample the subspace in the surroundings of selected stain by applying a pheromone trail with random Gaussian distribution. The probability that a stain occurs at x is determined by the following formula:

$$p(x) = g(x, \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}},$$

where the expected value $\mu = s_j^r$ is related to the position of the stain selected by ant, while the standard deviation $\sigma = \xi \sum_{p=1}^{k} \frac{|s_p^r - s_j^r|}{k-1}$ is the average distance between the r-th component of selected stain and the others within the population.

This construction is repeated for each of m ants (m new pheromone "stains" are obtained).

As it was mentioned, the modification of method for the multi-objective optimization is the replacement of the standard selection by the ranking one, in which the solution rank depends on belonging to a given layer of non-dominated solutions and the assumption that the succession operation keeps the best stains (solutions) in the iteration and "evaporates" the others.

4. Numerical results

In the tests that were designed to demonstrate the efficiency of the method, the test functions were optimized and the results were compared with the solutions available in the literature [13].

Additionally, two technical problems were solved in order to present the possibilities of using RACO in the optimization at the design stage. Finally, the comparison of solutions obtained by using an objective function for scalar optimization (aggregate function) and those obtained in the task of seeking full Pareto front were performed.

It was decided that in all examples, due to the better possibility of presentation and evaluation of results, the two-criteria functions are optimized.

For the purpose of each example calculations were performed in 10 different startups. All obtained results were comparable. In subsections below, one of the results is presented (any special method was used for selection of solutions for presentations).

4.1. Numerical tests – benchmark-type functions

Firstly, in order to demonstrate the proper functioning of both proposed methods and in-house software, the Pareto front of two test functions, ZDT1 and ZDT3, was searched [13, 14]. For both functions, the optimum solution is known, which allows to assess the quality of the result obtained in the tests.

4.1.1. ZDT1

The optimization problem of ZDT1 function is defined as

$$\begin{aligned} &\mathbf{f}\colon \mathbb{R}^{10} \longrightarrow \mathbb{R}^2, \\ &\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x})) \longrightarrow \min, \end{aligned} \tag{1}$$

where

$$\begin{aligned} &f_1(\mathbf{x}) = x_1, \\ &f_2(\mathbf{x}) = g(\mathbf{x}) \cdot \left(1 - \sqrt{f_1(\mathbf{x})/g(\mathbf{x})}\right) \end{aligned} \tag{2}$$

and

$$\begin{aligned} &g(\mathbf{x}) = 1 + \sum_{i=2}^{10} x_i, \\ &\mathbf{x} = (x_1, \cdots, x_{10}). \end{aligned}$$

The Pareto front (the set of optimal solutions) for the ZDT1 problem is obtained for $g(\mathbf{x}) = 1$, which means that it is a convex set $S_{ZDT1} = \{(x, 1 - \sqrt{x}) : x \in \mathbb{R}\}$.

The left graph in Fig. 2 shows the front of Pareto-optimal solutions (diamonds) obtained as a result of RACO's work on the background of the set S_{ZDT1} (solid curve). It is easy to notice a strong compatibility between the position and shape of the optimal solution and a set of non-dominated solutions obtained numerically.

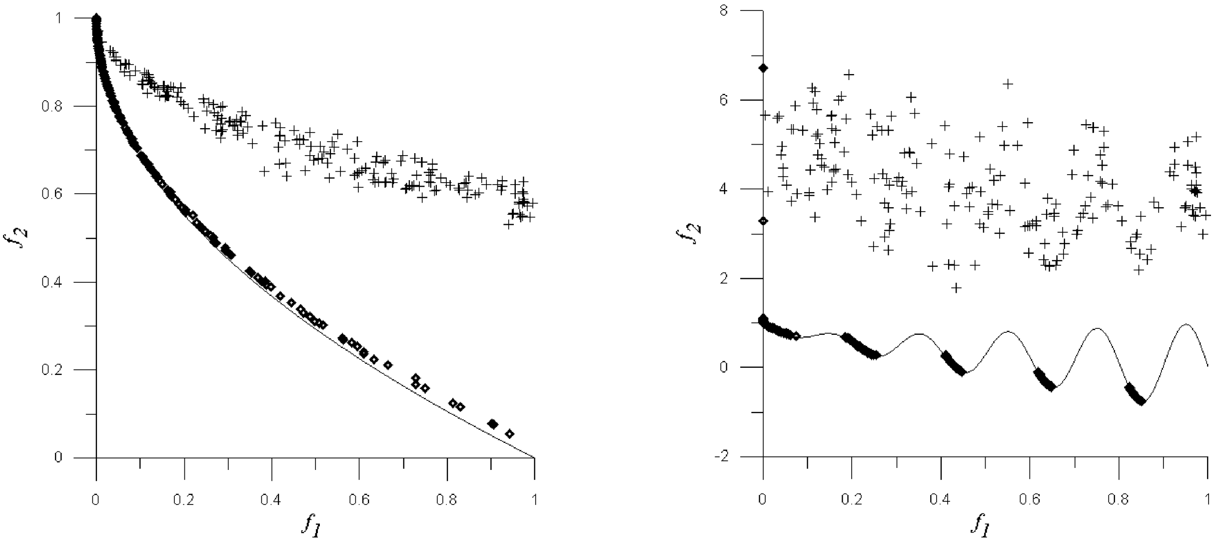

Fig. 2. The Pareto front at the background of initial population – ZDT1 (left), ZDT3 (rigth).

The set of points marked Fig. 2 by pluses is a set of values of the optimized function assigned for the initial population (randomly generated with a uniform distribution on $[0,1]^{10}$).

For the calculations, the population of 100 pheromone stains and 40 ants were used. The calculations were stopped after the number of iterations $maxIter = 100$. It means that $100 \cdot 100$ objective function evaluations were needed. After $maxIter$ iterations more than 95% pheromone stains belonged to set of non-dominated points and were treated as Pareto solution of stated problem.

The existence of analytical solutions gives in this case an additional opportunity to evaluate the accuracy of the resulting solution. For this purpose the average distance of non-dominated points and the actual front was tested. The 10 startups this distance does not exceed 0.01 what is comparable with the results available in the literature obtained by other heuristic methods [2, 12, 14].

4.1.2. ZDT3

In the second test, the function ZDT3 was optimized. It was defined, similarly to ZDT1, as

$$\begin{aligned} &\mathbf{f}\colon \mathbb{R}^{10} \longrightarrow \mathbb{R}^2, \\ &\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x})) \longrightarrow \min, \end{aligned} \tag{3}$$

except that

$$\begin{aligned} &f_1(\mathbf{x}) = x_1, \\ &f_2(\mathbf{x}) = g(\mathbf{x}) \cdot \Big(1 - \sqrt{f_1(\mathbf{x})/g(\mathbf{x})} - (f_1(\mathbf{x})/g(\mathbf{x})) \cdot \sin(10\pi f_1(\mathbf{x}))\Big), \end{aligned} \tag{4}$$

where again

$$\begin{aligned} &g(\mathbf{x}) = 1 + \sum_{i=2}^{10} x_i, \\ &\mathbf{x} = (x_1, \cdots, x_{10}). \end{aligned}$$

The Pareto front (the set of optimal solutions) for such formulated problem was also determined for $g(\mathbf{x}) = 1$. In contrast to the ZDT1 test, the Pareto front in this case is discontinuous and is the sum of non-dominated points of the set

$$S_{\mathrm{ZDT3}} = \{(x, 1 - \sqrt{x} - x\sin(10\pi x)) : x \in \mathbb{R}\}.$$

A comparison of the front obtained using the method proposed in this article and the analytical solution is presented in the right graph in Fig. 2. Also in this case, the solution obtained by RACO agrees very well with the analytical solution.

As in the previous example, in addition to the obtained Pareto front, the values of optimized function found for the initial population are also presented. It should be noted that the solid line represents the full set S_{ZDT3} while the solution to the problem is only a set of non-dominated points belonging to S_{ZDT3}.

Repeated calculations always give similar results, well matched to the reference Pareto front.

4.2. Numerical tests – technical problems

As the examples of engineering applications, two technical problems were solved. In the first one, dimensions (height and thickness of the wall) of a cantilever beam were optimized with respect to the weight and maximum stress reduction. The second problem concerned the minimization of stress and weight of a cryogenic pipe, partially filled with liquid and subjected to thermal load [2]. In this case, the thickness of the pipe wall and the liquid level were the decision variables.

The examples were chosen in order to determine, easily and with minimal computing cost, the values of criteria in optimized problem. In both cases, the criteria on which the solution was evaluated, were determined in analytical way as a function of the search parameters.

Of course, nothing precludes to exploit this method in the optimization of more complex technical problems. The only difference will be a longer time needed for assessment of the objective function, what means the determination of the values of all criteria.

The calculations 100 pheromone stains and 40 ants were used. The stop criterion was $maxIter$ = 100 iterations.

4.2.1. Beam optimization

In the first problem, the cantilever beam subjected to a constant load $\mathbf{F}$ and schematically shown in Fig. 3 is considered.

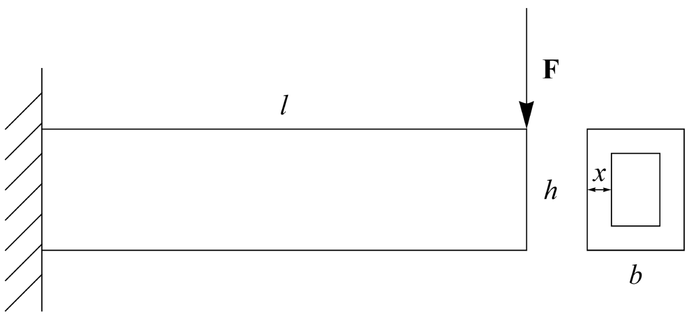

Fig. 3. Scheme of a beam example.

The goal of optimization is to minimize the maximum stresses σ caused by the load and the weight of the beam m_b. The decision variables are the height of the beam h and a wall thickness x at a fixed width b, the length L and the load value $\mathbf{F}$ (Fig. 3).

In the optimization problem it is assumed that the first criterion p_1 is the weight of the beam and the second p_2 – the maximum stress. It means that the following (objective) function was subjected to optimization:

$$\mathbf{f}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \qquad \mathbf{f}(x,h) = (m_b, \sigma) \longrightarrow \min .$$

The weight m_b based on beam geometry is defined as

$$m_b = \varrho \cdot l \cdot [b \cdot h - (b - 2x) \cdot (h - 2x)] ,$$

where ϱ is material density.

The maximum stress σ was determined in the analytical model:

$$\sigma = \frac{Flh}{2I},$$

where

$$I = \frac{bh^3}{12} - \frac{(b-2x)(h-2x)^3}{12}.$$

For the calculations, the following parameters were adopted: $l = 1$ m, $b = 0.2$ m, $\rho = 7800$ kg/m^3, $|\mathbf{F}| = F = 5000$ N. The Pareto front obtained in the calculations (solutions marked by diamonds) on the background of the function values obtained for the initial population (pluses), is presented in left graph in Fig. 5. Each of the resulting configurations is a Pareto-optimal solution, and the final choice, e.g., for implementation, must be undertaken by a human decision maker, after taking into account additional (non-numerical) criterion.

4.2.2. Cryogenic pipe optimization

The second problem concerned the minimization of stress $\sigma_{\max}$ and weight m_p of a cryogenic pipe, partially filled with liquid and subjected to thermal load [2]. In this case, the thickness of the pipe wall t (for fixed average diameter D_m) and the liquid level ξ were the decision variables (Fig. 4).

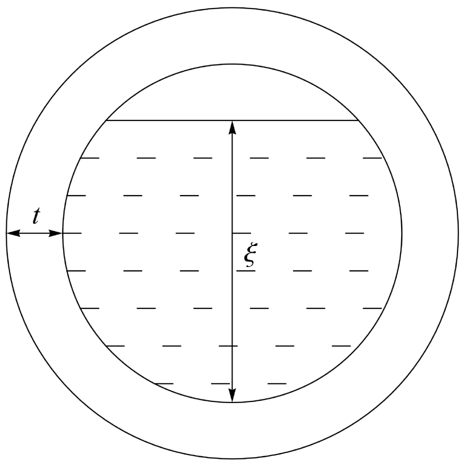

Fig. 4. Scheme of a cryogenic pipe example.

Therefore, the following function was subjected to optimization:

$$\mathbf{f}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \qquad \mathbf{f}(t,\xi) = (m_p, \sigma_{\max}) \longrightarrow \min.$$

As in the previous example, the pipe weight was determined based on the pipe geometry,

$$m_p = \varrho \cdot l \cdot \pi \cdot D_m \cdot t,$$

and the maximum thermal stress was determined in the analytical model according to the following formulation:

$$\sigma_{\max} = \alpha E \Theta_q \left[\frac{2}{3\pi} \arccos^3(2\xi - 1) + \frac{4}{\pi}(2\xi - 1)\left(2\sqrt{\xi(1-\xi)} - (2\xi - 1)\arccos(2\xi - 1)\right)\right],$$

where

$$\Theta_q = \frac{q(D_m + t)D_m}{8\lambda t}.$$

For the calculations, the following values were adopted $\rho = 7800$ kg/m^3, $l = 1$ m, $D_m = 0.2$ m, $\lambda = 12.3$ W/mK, $\alpha = 15e^{-6}$ K^{-1}, $E = 200$ GPa, $q = 470$ W/m^2.

The Pareto front obtained in the test is presented in the right chart in Fig. 5. The set of points marked by diamonds is a set of Pareto-optimal solutions found by RACO. All non-dominated points represent the solutions which are proposals of compromise and are "indistinguishable" without additional human decisions.

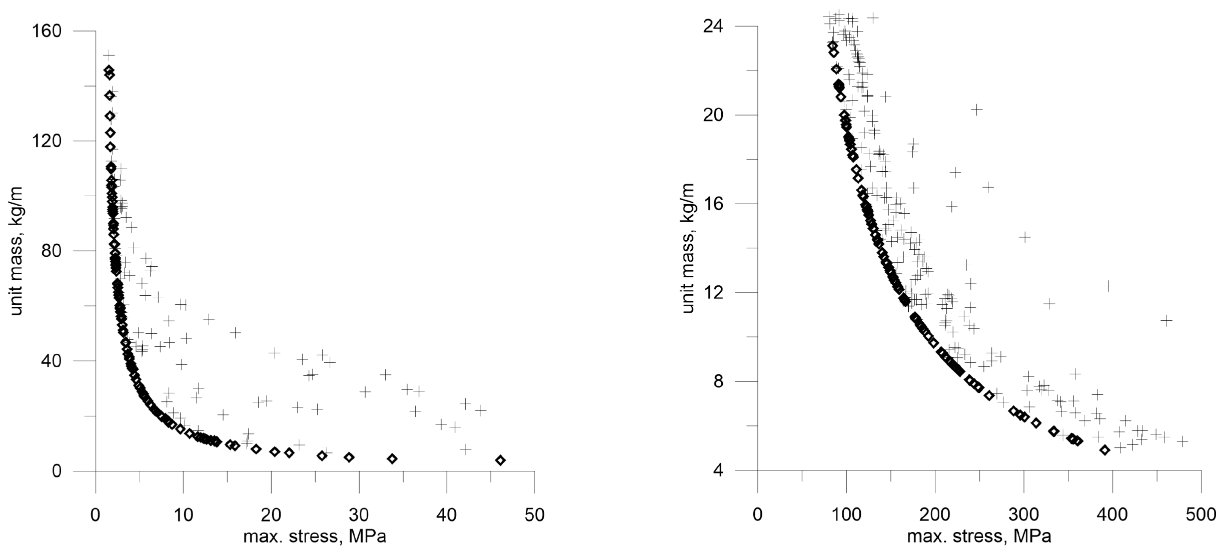

Fig. 5. The Pareto front at the background of initial population – beam optimization example (left), cryogenic pipe optimization (right).

4.3. Finding of Pareto front versus scalar function optimization

As a second test, the multi-criteria optimization problem was solved using both approaches: search before decision (used in the presented work) and decision before search (optimization of the aggregate function).

Both methods were compared based on the ZDT3 test function optimization.

The determining of (full) front Pareto, in search before decision option, was performed in a way described in Subsec. 4.1.2.

In the second variant, the weighted sum of criteria was chosen as an aggregate function φ_s:

$$\varphi_s(\mathbf{x}) = w_1 \cdot f_1(\mathbf{x}) + w_2 \cdot f_2(\mathbf{x}) \longrightarrow \min, \tag{5}$$

where criteria f_1, f_2 are defined as in (4).

Optimization of function (5) was carried out repeatedly. Weights w_1, w_2 were changed every time when the optimization procedure has begun. Thus, at least a few different solutions could be found (one for each start-up). It was assumed that w_1 varies from 0.1 to 0.9 with a step of 0.05, while $w_2 = 1 - w_1$.

The calculations gave the surprising results.

Optimization of scalar aggregate function for different values of the weights did not produce the solutions closely correlated with the weights. The solutions obtained in various start-ups are not uniformly distributed along the front of non-dominated solutions. Some of them are focused on two points, located near the ends of the Pareto front, while the position of the others is coincidental (see Fig. 6). Although this does not have to be a general principle applicable to any multi-objective function, it means that even multiple optimizations performed with different configurations of weight do not give the full information about the Pareto front. Weights controlling is not a linear process and, consequently, the researcher has no control of the direction of searches.

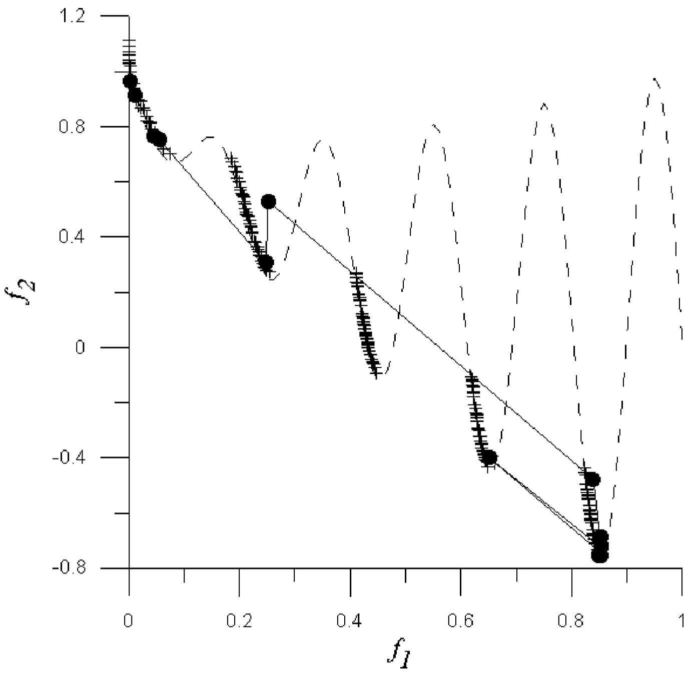

Fig. 6. Solutions of weighted sum of criteria (dots) and the Pareto front (pluses) at the background of analytical solution od ZDT3 problem.

The explanation may be the fact that in the iterative procedure, the current (working) point goes to a zone of attraction of one of the extreme optima. In the iterative process, the trajectory of a working point depends not only on weights but also on the starting point (initial population) and the randomness occurring in the optimization methods.

In turn, the search of Pareto front, in the variant of search before decision offers a wide range of solutions, which can be used in design of product, technology, etc.

It should be noted that for the approaches in which a full front of the non-dominated solution is searched, the number of evaluations for the objective function is usually higher in comparison with the methods using the optimization of scalar function. If finding of the single solution is satisfied from the point of view of the optimization goal, the optimization of scalar functions is sufficiently effective and efficient.

However, if there is a need to compare several/many variants of optimal Pareto-solutions, the methods for determining a set of as many as possible non-dominated solutions are much more competitive.

Similar observations were made by De Weck and described in work [11], where on the basis of the tests the following conclusions have been formulated. The scalarization approach is capable of finding the interesting solutions, but unfortunately:

1. Many interesting Pareto points are missed.
2. The resulting optima are unevenly distributed.

Nakayama in [8], has shown an example in which it was presented that it is usually very difficult to adjust the weight to obtain a solution desired by a decision maker.

Of course, the mentioned disadvantages could be bypassed by expansion of the multiattribute utility analysis (MAUA) methods [7], what requires the construction of the utility function and additional analysis.

5. Conclusions

A multi-objective (vector) optimization problem generally has more than one solution.

This paper proposes to use RACO, a heuristic inspired by behavior of ants' colony, for solving multi-objective problems.

The carried out tests have given satisfactory results, which show high efficacy of the method proposed in finding the front of Pareto-optimal solutions. RACO can be an alternative to the EA due to its more advanced model of space search. A probabilistic model of generating new proposals of solutions is more efficient than the scheme used in the EA [5].

Searching a large representation of points on the Pareto front offers a wide range of solutions which can be used in, e.g., the design of equipment, technology, etc.

In this paper, the results obtained in numerical tests (benchmark and technical ones) are presented. The carried out tests have given satisfactory results, which shows a promising efficacy of RACO in solving multi-criteria problems.

The comparison of solutions obtained in two variants of multi-objective optimization is also presented. As it is known, the optimization of aggregate scalar function is effective, but it must be remembered that the solution (single) depends on its definition determined at the beginning of optimization process. The calculations showed that "control" of weights is not linear and, as consequence, the researcher has no control over the direction of searches. This means that even in the case of multiple optimization performed for various combinations of aggregate function, the full information about the Pareto front is not obtained.

The number of single evaluations of the objective function is usually higher in the method searching the front of non-dominated solutions, but it should be remembered that cardinality of population has a direct impact on the cardinality representation of the Pareto front (its reduction is not appropriate from the point of view of the main purpose of multi-objective optimization).

In the future, it is planned to use the method presented for multi-criteria optimization in problems of engineering-intensive computing (parallelization of calculation is possible).

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