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Applications of Michell's Theory in Design of High-Rise Buildings, Large-Scale Roofs and Long-Span Bridges

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This paper analyzes the relations between the theory of Michell structures, which is one of the most important theories in structural optimization, and some remarkable engineering structures, including selected high-rise buildings, large-scale roof coverings and long-span bridges. The first part of this study briefly presents the development of Michell's theory, its basic concepts, assumptions, and examples and fundamental features of Michell structures. Then, several untypical engineering structures that make use of said concepts are presented, including skyscrapers proposed by the Polish structural designer W. Zalewski and the international architectural office of Skidmore, Owings and Merill (SOM). Next, large-scale roof coverings of "Spodek" arena in Poland as well as selected bridges are thoroughly analyzed in the context of similarity to Michell structures. The conducted study reveals that considered structural forms of the analyzed structures follow some of the concepts known from Michell's theory and thus possess many features of the optimal structural designs.

Keywords: topology optimization, Michell structures, high-rise buildings, large-scale roofs, long-span bridges.

1. Introduction

The theory of Michell structures is one of the most important and probably one of the most impressive theories in structural optimization. Michell's theory reveals how to optimally transmit the given external load to a given support and optimally transmit a given system of self-equilibrated loads. In contrast to traditional structural optimization settings, in which only selected parameters are considered as problem unknowns, in Michell's theory the entire structure is treated as a design variable. The results from such an approach allow drawing general conclusions about optimal layout, topology and geometry of the optimal structure, including members' connections, directions and sizing. Michell's theory discloses that bending in optimal structures is totally eliminated, while single members are fully stressed by tensile or compressive forces. As a result, the structure is perfectly adjusted to the applied external loading, requires a minimal amount of material and has minimal total weight.

Anthony George Maldon Michell proposed the above-mentioned pioneering concept in his remarkable paper published in 1904 [1], where, e.g., the problem of the optimal cantilever supported on a circle was solved. Since that time, many specific Michell structures involving various types of external loads (e.g., concentrated, distributed and transmissible) as well as various types of boundary conditions (e.g., roller and pinned supports) have been presented. The most significant contributions to the field were made in 1960s by H.S.Y. Chan who determined geometries of the optimal structures in bounded domains [2, 3], and in the books authored by Hemp [4] and Cox [5], where various types of elementary Michell structures were described. The interest in Michell's theory was revived in the 1990s by G.I.N. Rozvany and his co-workers, who analyzed problems regarding layouts of structures located in bounded design domains of various shapes [6, 7], various allowable stresses in tension and compression, multiple load cases and the supports costs [8, 9].

In recent decades, the approximate solutions of Michell's problems have been obtained numerically using *ground structure methods*, which rely on the selection of the optimal structure from the initially assumed system of nodes and their possible connections. The effectiveness of such methods results from the application of dual formulation of the weight minimization problem and linear programming methods, as well as the use of adaptive techniques limiting the number of simultaneously considered members, as described by Gilbert and Tyas [10] and Sokół [11]. The application of ground structures with millions of potential unknowns yields optimal designs that clearly resemble analytical layouts obtained from Michell's theory and gives almost exact values of corresponding volumes. As a result, in [12] the ground structure methods that allowed to find previously unknown Michell structures transmitting three self-equilibrated forces were presented, in [13, 14] structures transmitting distributed loading to two simple supports were studied, and [15] pointed out that selected known solutions of Michell's problem were entirely incorrect. The theory of Michell structures, as well as analytical and numerical solutions of the problems involving various loading and boundary conditions, is comprehensively presented in the book by Lewiński, Sokół and Graczykowski [16].

The mentioned above Michell structures were not only the theoretical concepts but also have inspired civil engineers and have influenced selected contemporary designs of high-rise buildings, large-scale roofs and long-span bridges. The objective of this paper is to present and critically analyze practical applications of Michell's theory in civil engineering. Firstly, we will study the concepts of Michell-inspired "wingy" and "bulbous" skyscrapers proposed by Wacław Zalewski and Wojciech Zabłocki as well as selected buildings designed by the international architectural office of SOM. Then, the constructions of the large-scale roofs of two famous Polish commercial buildings, Supersam in Warsaw and Spodek in Katowice, will be analyzed, revealing that their designers used a combination of the tensegrity concept and Michell's theory. Finally, conclusions from Michell's theory concerning the optimal layout of structures created over multiple spans and subjected to distributed loading will be compared against selected constructions of long-span bridges.

2. Theory of Michell structures in a nutshell and illustrative examples

2.1. Michell structures in the plane

The topology optimization problem considered in the theory of Michell structures is to find the lightest structure with the bounded value of stress: $-\sigma_C \leq \min \lambda_i(\boldsymbol{\sigma}) \leq \max \lambda_i(\boldsymbol{\sigma}) \leq \sigma_T$, which transmits a given load to a given support or transmits a system of self-equilibrated loads (Fig. 1); $\lambda_i(\boldsymbol{\sigma})$ represents the i-th eigenvalue of tensor $\boldsymbol{\sigma}$.

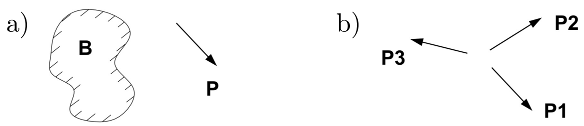


Fig. 1. Two versions of 2D Michell's problem: a) transmitting a given point load to a given support, b) transmitting a self-equilibrated system of loads.

The optimal structures should be examined in the class of trusses since the elimination of bending leads to a uniform state of stress in the entire cross-section of each member, while optimization of member's cross-sections allows to obtain the fully-stressed design of the entire structure. The structures proposed by A.G.M. Michell are a generalization of trusses and they take the form of discrete-continuous structures composed of:

- *fibrous domains* consisting of infinitely thin and infinitely densely located members (a single family of straight fibers or two families of orthogonal fibers),

- *reinforcing members* of finite cross-sections (typically located at the boundaries of the feasible domain) being the response for the occurrence of point loads.

In the non-degenerated case, when two families of members exist, all members are located along the trajectories of principal strains. The theory of Michell structures originated by solving the problem of finding the lightest cantilever of equal permissible stresses in tension and compression capable of transmitting a given point load to a given circular support. The approximation of this solution with a finite density of parametric lines is presented in Fig. 2 (left).

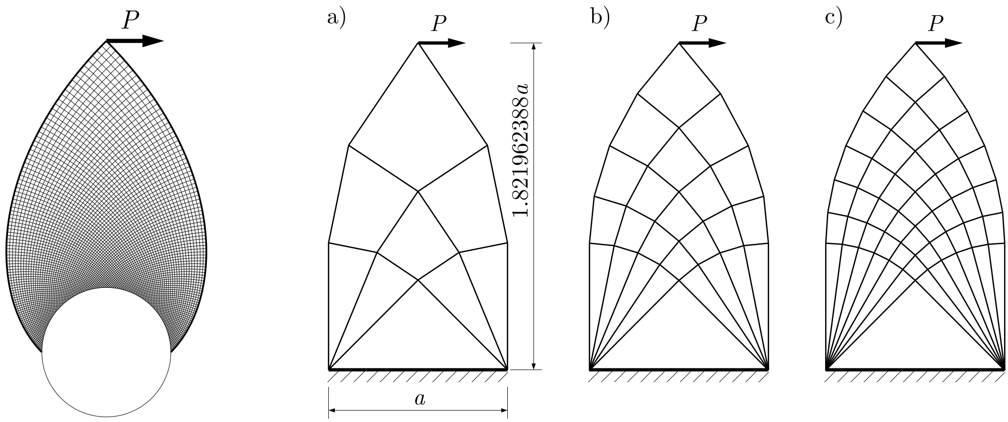


FIG. 2. Left: Michell cantilever transmitting point load to circular support (case of equal permissible stresses in tension and compression); right: three subsequent suboptimal trusses transmitting given force to a straight support: a) $V_r = 6.1605$, b) $V_r = 6.0953$, c) $V_r = 6.0835$, where $V_r = V/V_0$, $V_0 = Pa/\sigma_0$, σ_0 being the permissible stress.

The intuitive understanding of Michell structures can be gained by analyzing three suboptimal trusses with an increasing number of members, transmitting a point load to a straight support (Fig. 2 right). These structures can be obtained using an arbitrary numerical method (e.g., ground structure method) with various discretizations within the planar design domain. Application of the numerical methods shows that when the spatial discretization becomes finer, the number of bars increases and the structure becomes lighter. The limit of this sequence corresponding to the infinitely dense discretization in the plane and characterized by infinite number of bars is a Michell truss. Although the structure presented in Fig. 2c is not infinitely dense, it reveals basic features of the corresponding Michell structure. It is composed of fan regions with straight members starting at the supports, fibrous domain with orthogonal members and reinforcing members of larger cross-sections (not visualized) spanning from the location of the point load to the locations of the supports.

Mathematical formulation of Michell's problem requires defining $U\left(\overline{\Omega}\right)$ as the set of kinematically admissible virtual displacements $\overline{\mathbf{u}}$, as well as the set $\Sigma\left(\overline{\Omega}\right)$ of statically admissible forces $\widetilde{\mathbf{N}}$, $\widetilde{F}_T$, $\widetilde{F}_C$ in fibrous domain and in both reinforcing members (along the curves Γ_T and Γ_C respectively) compatible with external loading. The principal forces in the fibrous domain can be defined as: $N_1 = h\lambda_1(\boldsymbol{\sigma})$ and $N_2 = h\lambda_2(\boldsymbol{\sigma})$, h being the depth (or the transverse thickness) of the structure. Thus, the total volume of the structure equals:

$$V_\Omega = I\left(\mathbf{N}, F_T, F_C; \overline{\Omega}\right) = \int_\Omega \left(\frac{|N_1|}{\sigma_T} + \frac{|N_2|}{\sigma_C}\right) d\overline{\Omega} + \int_{\Gamma_T} \frac{|F_T|}{\sigma_T} ds + \int_{\Gamma_C} \frac{|F_C|}{\sigma_C} ds \quad (1)$$

and the primary formulation of the volume minimization problem reads:

$$V_\Omega = \min\left\{I\left(\widetilde{\mathbf{N}}, \widetilde{F}_T, \widetilde{F}_C; \overline{\Omega}\right) \text{ such that } \left(\widetilde{\mathbf{N}}, \widetilde{F}_T, \widetilde{F}_C\right) \in \Sigma\left(\overline{\Omega}\right)\right\}. \quad (2)$$

The dual formulation takes the form of maximization of the work of the external forces on virtual displacements:

$$V_\Omega = \frac{1}{\sigma_0} \max\left\{\mathbf{P} \cdot \overline{\mathbf{u}}\,(\mathrm{P}) \;\middle|\; \text{such that } \overline{\mathbf{u}} \in U\left(\overline{\Omega}\right);\ \boldsymbol{\varepsilon}\left(\overline{\mathbf{u}}\right) \in B_\kappa\right\}, \quad (3)$$

where B_κ is the so-called locking locus confining allowable values of virtual strains to those satisfying:

$$-\frac{\sigma_0}{\sigma_C} \leq \lambda_i(\boldsymbol{\varepsilon}(\overline{\mathbf{u}}) \leq \frac{\sigma_0}{\sigma_T}, \qquad i = 1, 2. \quad (4)$$

The above formulation allows to determine the optimality conditions which read:

$$\lambda_1\left(\boldsymbol{\varepsilon}(\overline{\mathbf{u}})\right) = \frac{\sigma_0}{\sigma_T}, \qquad \lambda_2\left(\boldsymbol{\varepsilon}(\overline{\mathbf{u}})\right) = -\frac{\sigma_0}{\sigma_C} \quad (5)$$

in the regions where two families of bars occur.

The important conclusion is that all members of the optimal structure have to be located along the trajectories of principal strains, where the strains achieve limit constant values. These trajectories may be curved so members of the optimal structure also have to be curved. As a consequence, there exist two families of fibers, which create an infinitely dense orthogonal net, the so-called Hencky net.

In order to find the Hencky net, we have to determine the Lamé functions $A(\alpha, \beta)$, $B(\alpha, \beta)$ defined as:

$$A(\alpha, \beta) = \sqrt{a_{11}}, \qquad B(\alpha, \beta) = \sqrt{a_{22}}, \qquad a_{\lambda\mu} = \mathbf{a}_\lambda \cdot \mathbf{a}_\mu,$$

where $\mathbf{a}_\lambda$ are bases vectors. The Lamé fields are governed by the differential equations:

$$\frac{\partial^2 A}{\partial\alpha\partial\beta} = A, \qquad \frac{\partial^2 B}{\partial\alpha\partial\beta} = B. \tag{6}$$

The following step is the construction of the mappings: $x(\alpha,\beta)$, $y(\alpha,\beta)$ defining the parametric lines and the adjoint displacement field $\overline{\mathbf{u}} = [u(\alpha,\beta),\ v(\alpha,\beta)]$. These fields have to satisfy the so-called telegraphers equation and kinematic boundary conditions. They can be found via the application of Riemann's method. Let us note that the above relatively complex procedure allows only for finding the geometry of Michell structure. The complete solution of Michell's problem is constructed in the following steps:

1) Finding geometry of the Hencky net for given support geometry and force location;
2) Finding force fields in fibrous domains and reinforcing members;
3) Finding the equivalent thickness field of the structure;
4) Computing the volume using: i) virtual work, see (3) and ii) integration of thickness, see (1).

The agreement of volumes obtained using the two above methods proves that the duality gap between the primal and dual formulation of the volume minimization problem vanishes and confirms the determined continuous-discrete structure's optimality.

The load-carrying engineering structures are usually designed within a certain feasible region. In many cases, its boundaries are segments of straight lines. Just the presence of the boundaries brings about specific shapes of the optimal solutions. Let us recall that the Michell cantilevers transmitting point loads into a straight support located in a trapezoidal domain are composed of several regions of the kinematic division with different geometry of parametric lines. The arrangement of these regions and parametric lines geometry do not depend upon the location of the point load (Fig. 3a). The initial straight support appears to degenerate into two pin supports, making the optimal structure externally statically indeterminate and implying a dependence of the geometry of parametric lines on allowable stresses in tension and compression (Figs 3a and 3b). The location of point load defines regions of the static division with a different distribution of internal forces (Fig. 3c). The distribution of the thickness of the cantilever is determined region-wise, and it depends both on the Lamé fields and the force fields inside the fibrous domain. The overlapping of the regions of kinematic and static division causes the cantilever thickness's distribution to be very complex and containing multiple lines of discontinuities (Fig. 3d). Discrete versions of Michell structures can be obtained using the method of graphic statics [17].

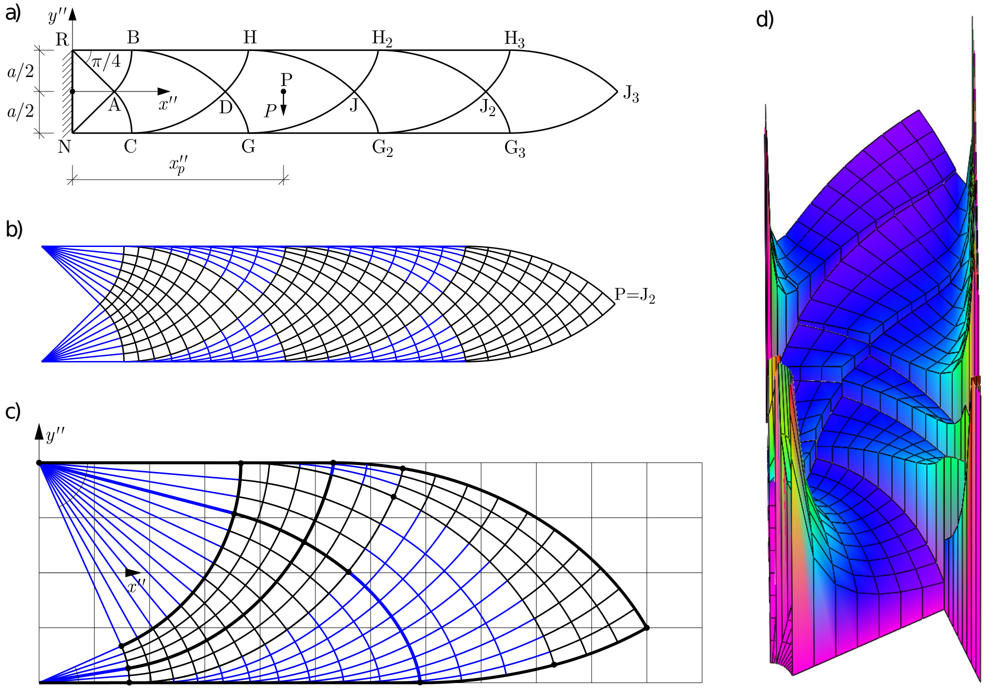


FIG. 3. Optimal Michell cantilever transmitting point load to straight support: a) arrangement of the regions of kinematic division for $\sigma_T = \sigma_C$; b) geometry of Hencky net for $\sigma_T = \sigma_C$; c) arrangement of regions of kinematic and static division for $\sigma_T = 5\sigma_C$; d) the alternate thickness distribution of the cantilever for $\sigma_T = 5\sigma_C$.

2.2. 3D setting: spatial Michell structures

The subject of consideration is the minimization of the volume of spatial frameworks to be designed in a given spatial domain; the state of stress is subject to the conditions: $-\sigma_C \leq \min \lambda_i(\boldsymbol{\sigma}) \leq \max \lambda_i(\boldsymbol{\sigma}) \leq \sigma_T$, $i = 1, 2, 3$. Since optimization excludes bending, the above conditions mean that the axial stresses in the bars (or fibers) are bounded by $-\sigma_C$, σ_T. The load is given and should be transmitted to a prescribed boundary where the supports can be placed; indeed, the position of supports is also determined by the optimization process. This setting includes, in particular, designing roofs over large base domains; the design domain is then a certain layer between two fixed surfaces over the basis domain.

As in the 2D case, the 3D design process is reduced to solving the two mutually dual problems. The kinematic problem has the form (3, 4) where now the locking locus for virtual strains is a cube. In contrast to the 2D case, its vertices need not be attained. Some subdomains may be characterized by virtual strains whose only two principal strains attain the bounds. Then the optimal structure becomes a grid surface. The dual or stress-based problem is expressed by Eq. (3.95) in Lewiński *et al.* [16]; hence it is reduced to finding a minimum of a certain functional of the statically admissible stresses.

The number of available exact 3D solutions to the problem stated above is very limited, see Chapter 5 in Lewiński *et al.* [16]. On the other hand, the ground structure method is still being improved to produce clear numerical solutions in 3D, see also Sokół [18]. New highly accurate numerical solutions of 3D problems will appear soon.

If the applied load is assumed to be transmissible along the gravity direction and if the roof designed over a given planar basis is to be composed of two families of mutually orthogonal arches, the optimum roof becomes a Prager-structure, see also Rozvany and Prager [19]. The theory of such roofs has much in common with the theory of Michell structures, since the optimization problem is also here reduced to the two mutually dual problems (kinematic and static) whose solutions determine the structure itself; the method cuts out the sub-domains of the basis domain over which the roof is not necessary, see Czubacki and Lewiński [20].

3. Applications to high-rise buildings design

The engineers who were strongly impressed by Michell's theory and inspired by his theoretical optimal layouts were Polish construction engineer and designer Wacław Zalewski (1917–2016) and architect Wojciech Zabłocki (b. 1930). During their long-term and fruitful cooperation, they have considered, among others, buildings taller than 200 m with height to width ratios larger than five ($h > 5a$), subjected to large torsional and bending moments caused by wind loads. Consequently, in order to develop the basic structural model of the building, they applied the analogous optimum design problem of the cantilever. They considered various standard structural systems and topologies based on variations of Michell cantilevers as candidate solutions for high-rise buildings design [21–23]. Moreover, they intuitively applied optimal structure rationalization by reducing it to trusses composed of dozens of elements, see also Allen and Zalewski [24]. The fundamental comparison of normalized volumes of these structures (Fig. 4a) has immediately shown the superiority of Michell-like topologies and revealed their usefulness in high-rise building construction. This encouraged Zalewski and Zabłocki to propose the general concept and develop several detailed projects of "tulip-like (bulbous) buildings" and "wingy buildings", presented in Figs 4b and 4c, respectively.

The "wingy buildings" are composed of three or four wings (which stand for buttresses) connected to the central core. The main feature of the central

a) b) c)

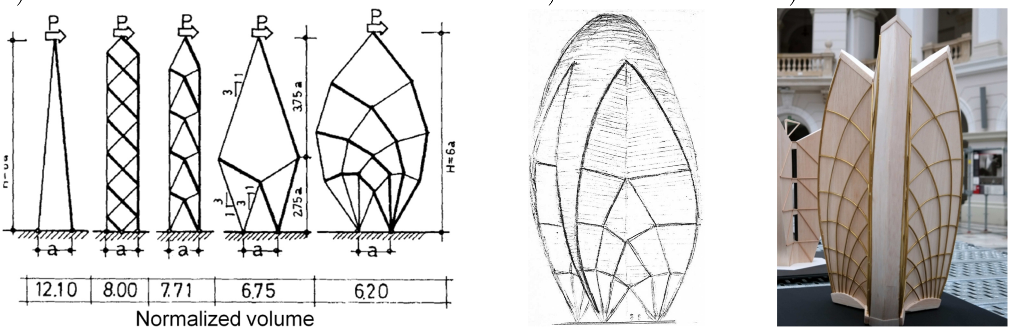


Fig. 4. a) Comparison of various topologies by Zalewski and Zabłocki [21]; b) sketch of "tulip-like building" (source: http://www.wz-structure.org/#/unbuilt/); c) wooden model of "wingy building" (source: https://www.pw.edu.pl/Uczelnia/Materialy-promocyjne/Galeria/Konstrukcje-Waclawa-Zalewskiego-fotorelacja).

core is the capability of resisting vertical loads caused by self-weight. Wings construction is based on a system of orthogonal members located at the exterior surfaces and thus possesses the ability to transmit bending moments caused by wind loads, which is also characteristic for Michell cantilevers. These innovative designs resulted in an attractive shape of the building and good lighting of spaces. In addition, the atypical aerodynamics of the building allows for wind energy harvesting.

The wingy skyscraper presented in Figs 5a and 5b has tapered wings connected to the central transit core. The building has 209 m in height and 50 stories. The self-weight is transmitted by vertical columns located in the central core. The structure of each wing clearly resembles the geometry of Michell cantilever. The members in tension and compression are located approximately orthogonally and roughly along the directions of principal stresses caused by bending forces. In addition, the round concrete foundations resemble the ends of the fan regions of the optimal cantilevers.

The "bulbous" ("tulip-like") skyscraper is presented in Figs 5c and 5d. The building has a bulbous shape and it is built on a plan of a clover. The structure has 164 m in height and 41 floors. The shape of the building is created by rotating a 2D Michell cantilever around the vertical axis of symmetry and by varying the slenderness. The vertical loads caused by self-weight are transferred by the central communication core. The double-curvature steel structure on the facade surrounds the entire building and transmits lateral forces caused by wind loads. Because of symmetrical construction, the building has equal resistance to wind loads from all possible directions.

The application of Michell's theory to buildings' construction can also be clearly seen in some selected designs by the architectural office of Skidmore, Ow-

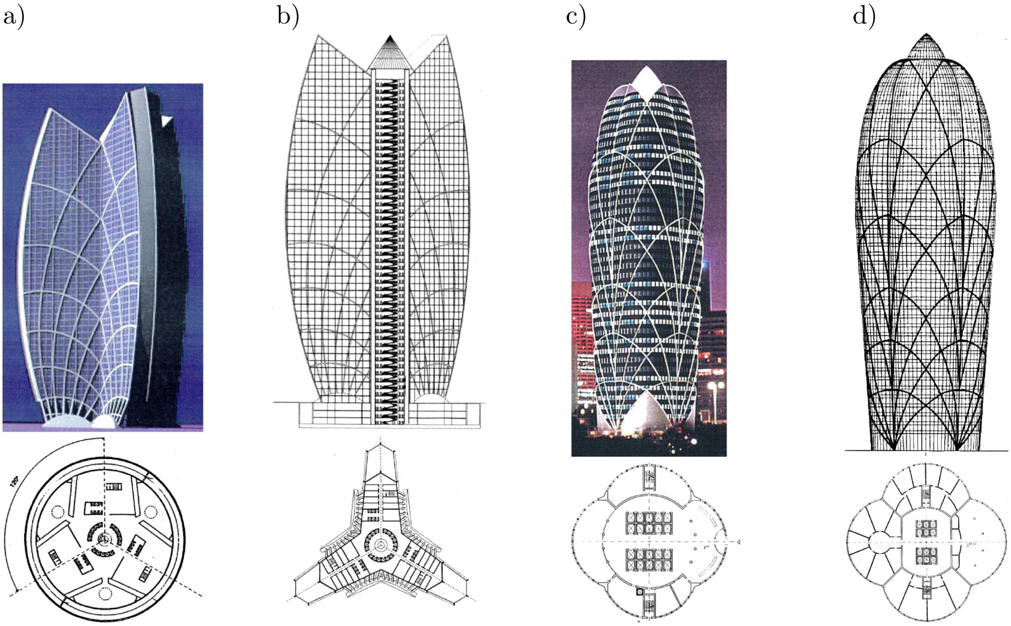


Fig. 5. The projects of buildings by W. Zalewski and W. Zabłocki [21]:
a, b) the project of "wingy skyscrapers"; c, d) the project of "bulbous skyscrapers".

ings and Merrill (SOM). One of the most illustrative examples is the Broadgate Exchange House in London designed by Srinivasa "Hal" Iyengar and William F. Baker (Fig. 6). The structure spans over 78 m long rail yard, which imposes the functional requirements of a building and a bridge. The structure is supported and strengthened by two large-scale arches located at both sides of the building. Multiple vertical members are used to transfer the building's weight to

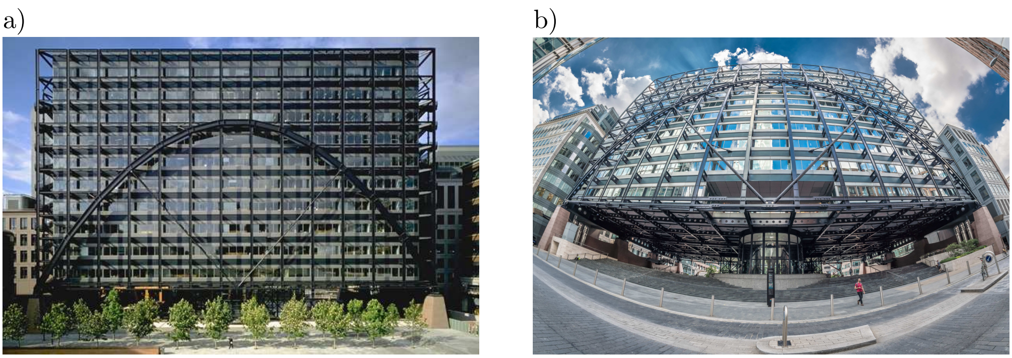
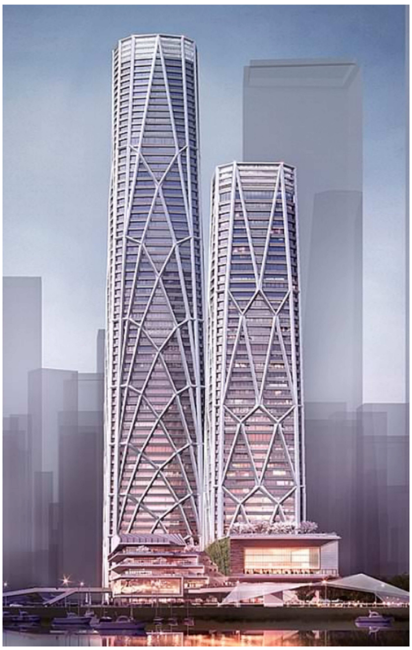


Fig. 6. Broadgate Exchange House in London (design: SOM: Srinivasa "Hal" Iyengar, William F. Baker): a) view of building front (source: https://commons.wikimedia.org/wiki/File: Broadgate_-_Exchange_House_BGATEEX_hero.jpg); b) fisheye view (source: https://www.flickr.com/photos/mhx/20535205995).

the arch, which transfers it further to the lateral supports. Two skew members are used to connect the arch and central point of the structure bottom and to facilitate the transfer of lateral loads. Let us note that in this case the multiple skew hangers (as presented, e.g., in Hemp's structure, see also the last result of Fig. 4.173 in [16]) are not required since there is no concentrated load. In addition, the shape of the arch is optimized in order to provide minimal weight and maximal stiffness using the Maxwell load path theorem.

Another design by SOM that is inspired by Michell's theory is the project of CITIC Financial Centre in Shenzen (Fig. 7). The complex is composed of two high towers – 200 m and 300 m, with external bracings. The shorter tower's structure is based on a very simplified layout of Michell cantilever with only two members at each level. This layout is doubled at the width and stretched in vertical direction (Fig. 7a). In turn, the taller tower has a simplified layout of Michell cantilever with four parametric lines at each level. The external bracing evidently possesses fundamental geometrical features of Michell cantilever. In particular, we can clearly observe the members fanning out at both sides of the building (Fig. 7b) and bottom circular fans connecting at the right angle (Fig. 7c). The vertical stretch resulting in changes of members' angles along

a)



b)

c)

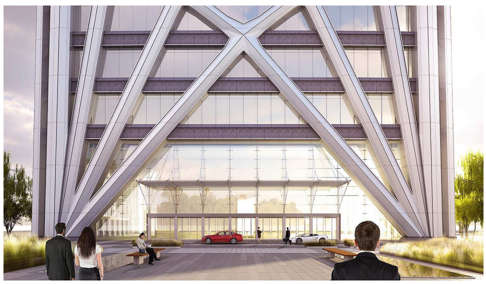


Fig. 7. The project of CITIC Financial Centre by SOM (Craig W. Hartman, courtesy of SOM): a) general view of both towers; b) detailed view of higher tower top; c) detailed view of higher tower bottom (source: https://www.som.com/projects/citic_financial_center).

the height of the building, combined with the usage of compliant joints, causes an improved global seismic performance of the building. The above buildings designed by SOM follow a general concept of using topology optimization to advance the process of building design [25].

4. Applications to large-scale roof designs

The applications of the elements of Michell's theory to design of large scale roofs can be clearly observed in constructions developed by Wacław Zalewski. The characteristic feature of his design method is shaping the structures in such a way that they are perfectly suited for transferring applied static loading. In particular, in Zalewski's design, the main load-bearing members are always dominated by compression or tension, while bending is eliminated to the largest possible extent.

The example of a design that follows this principle is a large-scale roof of the self-service store "Supersam" in Warsaw designed by W. Zalewski, S. Kuś and A. Żórawski, and erected in 1962. The roof structure is supported by alternately located parallel convex arch steel girders and concave prestressed cables of parabolic shapes (Fig. 8). By proper shaping of the steel elements, the designers obtained almost constant values of axial forces along the span of the roof, which facilitated the optimal selection of elements' cross-sections and enabled minimization of material usage.

Fig. 8. The roof of the Supersam in Warsaw (source: https://pl.wikipedia.org/wiki/Supersam#/media/Plik:SuperSam_Warszawa_1969.jpg).

Moreover, the application of such construction allowed to obtain a perfect interplay of tensile and compressive forces. In particular, at the side columns, the outward component of the arches' thrusts equalizes the inward components of the cables' pull such that the columns are not subjected to bending moment and they can be designed quite slender. A more detailed description of the Supersam roof design can be found in [16] and publications by Zalewski [26, 27].

Another remarkable structure designed by Wacław Zalewski is the event hall "Spodek" (English: "Saucer"), whose design combines the elements of Michell's theory and the tensegrity concept [28, 29]. Spodek is a multipurpose arena for 11 500 spectators, formed in an extremely untypical way as a flying saucer, an iconic shape of UFO (Fig. 9). It is the first arena in Poland built based on the tensegrity principle, and still remains one of the largest of this type. The erection of the structure was started in 1964, and it was stopped for almost two years due to suspicion of alleged construction errors. After verifications of static calculations by independent experts, it was successfully completed in 1971. Spodek hosts many international cultural and sport events and remains the landmark of the city of Katowice until now.

a)

b)

Fig. 9. "Spodek" multipurpose arena in Katowice: a) general view (source: https://pl.wikipedia.org/wiki/Spodek_(hala_widowiskowa)#/media/Plik:Hala_Spodek_2007-10.jpg); b) view of the roof with central skylight (source: https://www.pexels.com/photo/katowice-ray-silesia-spodek-84918/).

From the architectural point of view, the main challenge in the design of Spodek was to accommodate both: sport events with a surrounding audience and music concerts with a directional audience. Atypical geological conditions, resulting in severe difficulties in foundation settlement, called for innovative solutions. The initial candidate location in the city suburbs was characterized by soft and weak soils, while the final location in the city center was a mining damaged area (waste dump site) with possible sinkholes, ground discontinuities, non-uniform soil subsidence and soil creep.

As a solution to this challenging engineering problem, Zalewski proposed an innovative asymmetric structural form resembling an inverted bowl or dome (Fig. 10), which provided the possibility for convenient adaptation to various types of cultural and sport events. The structure has a spectator stand surrounding the central part of the hall, which is strongly asymmetric and significantly widens in the upward direction. The supporting structure for the stand has the shape of the inverted cone. In this structure, the circumferential tension is counterbalanced by prestressing, while the weight of the roof and the stands is transmitted to a foundation ring of relatively small dimensions. Zalewski carefully considered the problem of difficult geological conditions. For the initial location with the weak ground, he proposed a shell foundation with a box-like bottom surface resembling an inverted bowl, effectively closing the central cone from the bottom and denting into soft soil. In turn, for the final location at the mining damaged area, he proposed an entirely different type of foundation – the circular ring stiffened by the horizontal diaphragm, which allows the entire structure to slide with respect to the creeping ground and settle uniformly as a rigid object.

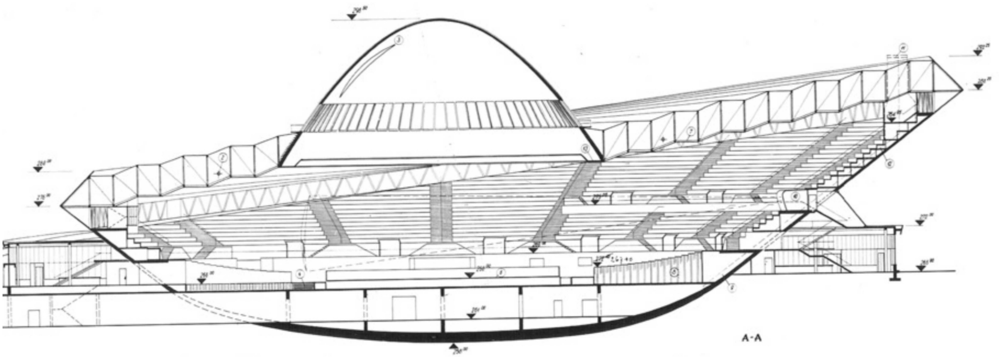


Fig. 10. Architectural drawing of the initial design of Spodek by W. Zalewski (source: http://www.wz-structure.org/#/spodek/).

Another important challenge was to design the large-scale roof that would cover the entire arena. The main problem in such large-scale roof design is the requirement of transmitting substantial radial and circumferential bending moments caused by the roof covering and external loading such as wind or snow. Three basic types of systems can be applied for large-scale roofs design. They are as follows [30]:

- The radial system in which radial bending moments are transferred directly by a system of radial beams connected at the central point. The system is characterized by the occurrence of large bending moments.
- The circumferential system in which the circumferential bending moments are transferred directly by concentrically located pairs of upper rings in compression and lower rings in tension connected by distance posts. The rings are strengthened by skew hangers (Fig. 11a) or supported by skew struts located in the opposite direction.
- The radial-circumferential second system in which both the circumferential and radial bending moments are transferred directly by the pairs of concentric rings and radially located straight members. In such a system, the radial compression of the upper elements causes additional bending in the circumferential direction.

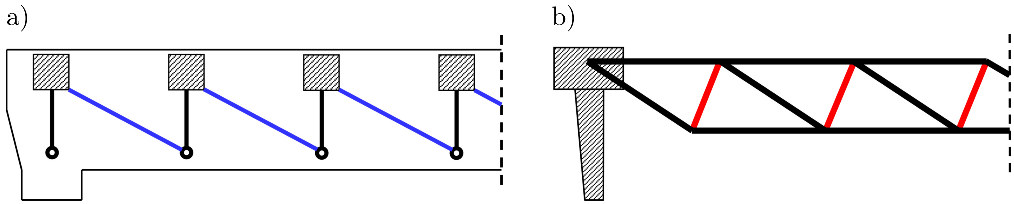


Fig. 11. a) The initial design of the Spodek roof by Wacław Zalewski, b) the final design of the Spodek roof (based on drawings by M. Pelczarski [30]).

Since the design aimed at the minimization of bending, Zalewski chose a circumferential system for the roof design, which was an extraordinary solution considering the scale of the building. By conducting detailed calculations, he proved that a larger amount of materials were not required compared to radial, radial-circumferential and other analyzed systems. Unfortunately, before the structure was erected, and after Zalewski emigrated from Poland, his concept of roof design was significantly modified, probably due to difficulties associated with the elliptical roof shape. Eventually, the constructors used 120 conventional 2D cable-strut girders with inclined posts (Fig. 11b), which was a proven concept known from industrial constructions.

Let us note that the entire construction of Spodek, including the inverted dome and the large-scale roof, can be considered a tensegrity system. Similarly, as in the case of Supersam, the structure effectively utilizes the balance of tensile and compressive forces. The inverted dome, which constitutes the support for the stage and the grandstands, is subjected to significant circumferential tensile forces caused by its self-weight and the weight of spectators. According to the original concept of Zalewski, these tensile forces are balanced by a prestressed steel ring located at the top of the dome. This ring is compressed by tensioned cables spanned orthogonally to the internal ring located in the center of the roof (Fig. 12). Once again, the perfect balance of tensile and compressive forces and

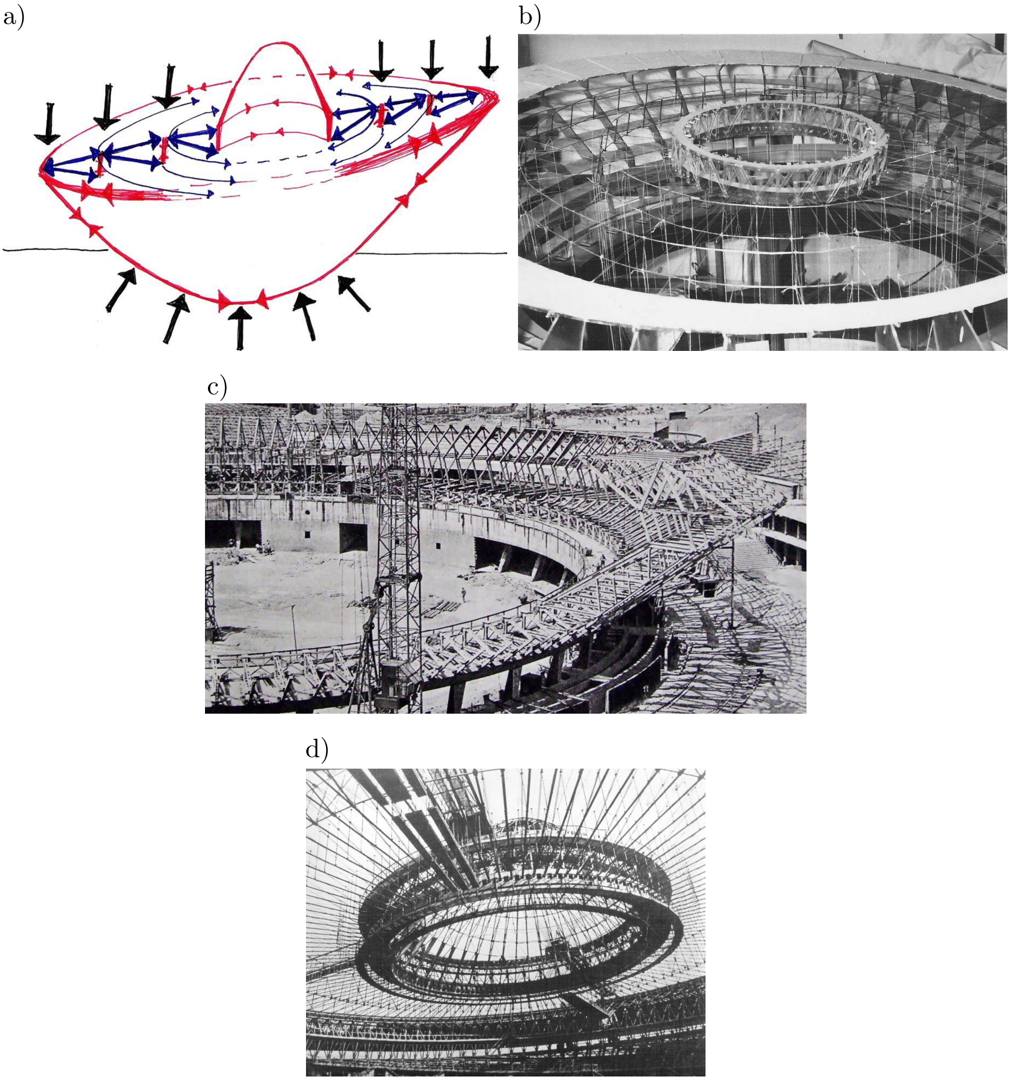


Fig. 12. a) Force diagram by E. Allen and D.M. Foxe (source: http://www.wz-structure.org/#/spodek/); b) scaled wired model used by W. Zalewski for structural concept testing (source: https://structurae.net/en/structures/spodek); c) external ring at the top of the bowl (source: https://fotopolska.eu/195932,foto.html); d) internal ring at the center of the roof suspended by tensioned cables (source: https://structurae.net/en/structures/spodek).

elimination of bending appear to be the most essential features of Zalewski's design.

In summary, it can be stated that "Spodek" ingeniously combines the tensegrity principle and Michell's theory. The elements of the tensegrity principle can be observed in the original Zalewski's design of the large-scale roof and the concept of prestressed ring balanced by tensioned cables. In turn, the correlation to Michell structures is revealed in the original shaping of the structure aimed at obtaining the desired distribution of internal forces as well as using orthogonal members transferring compressive and tensile forces.

5. Applications to long-span bridges design

The Michell theory has also had an impact on the art of the optimal shaping of bridges. Probably the most important results concern designing for a uniform load over multiple spans [14], and optimal forms of very long-span bridges under gravity loading [31]. The solution for the uniform load reveals that the topology of the optimal structure strongly depends on the ratio of allowable stresses in tension and compression. In the case of significantly larger allowable compressive stresses, the optimal structure constructed over a single span resembles a modified design of the arch bridge; it contains a compressed arch, nearly vertical hangers and reinforcement extending from the supports. By contrast, in the case of significantly larger allowable tensile stresses, the optimal structure resembles a modified design of a cable-stayed bridge; it is composed of posts inclined toward the center of the span and long cables connecting the posts and the bridge deck.

In turn, the solution of the problem involving self-weight reveals that independently of the length of the bridge, the optimal structure contains pylons split at an obtuse angle and hangers spanning to the deck. In [31] such optimal structure is compared against less efficient structures: i) optimized triple split-pylon design with one vertical and two skew members, ii) optimized double split-pylon design with two skew members (approximation of the optimal designs), iii) optimized cable-stayed design and iv) optimized suspension bridge design. The increase in the volume of the subsequent structures appears to increase with the length of the bridge. In particular, the increase in the volume of double split-pylon design compared to optimal design is 1.07 and 1.12 for span lengths of 1 km and 5 km, respectively. The analogous coefficients for the suspended bridge equal 1.40 and 1.73. In addition, the above study reveals that the design of the world longest suspension and cable-stayed bridges: Akashi Kaikyo (span: 1991 m) and Russky Bridge (span: 1104 m) is highly non-optimal since pylons height should be significantly increased.

An interesting type of bridge design, partially based on the above results from Michell's theory, combines the concept of a cable-stayed bridge with pylons and oblique cables with the concept of an arch bridge with vertical hangers. An illustrative example of such structure is Seri Saujana Bridge in Putrajaya (Malaysia), whose construction is based on inclined pylons and a steel arch. The cables attached to the pylons support the middle of the deck, while hangers attached to the arch support its edges (Fig. 13a). A similar design was used in the Lianxiang Bridge in Hunan (China). In this case, the construction successfully utilized straight H-shaped pylons and an arch constructed as a spatial truss (Fig. 13b).

a)

b)

Fig. 13. Bridges combing cable-stayed and arch type of construction: a) Seri Saujana Bridge in Malaysia (source: https://en.wikipedia.org/wiki/Seri_Saujana_Bridge#/media/File:Seri_Saujana_Bridge_2009. jpg), b) Lianxiang Bridge in China (source: https://commons.wikimedia.org/wiki/File:Lianxiang_bridge.jpg).

The designs following the above-presented concept of pylons splitting can be found both among long-span road bridges and short-span pedestrian bridges. An illustrative example of this is the split-pylon bridge "Flughafenbrűcke" near Dűsseldorf airport (Fig. 14a). In this case, the designers' objective was to modify the classical construction of a cable-stayed bridge avoiding straight pylons due to proximity of the airport and expected air traffic. For the required length of the span of 287 m, the required total height of the straight pylon was 110 m. The a)

b)

Fig. 14. Untypical bridges with double split-pylon: a) Flughafenbrücke near Düsseldorf airport (source: https://commons.wikimedia.org/wiki/File:Flughafenbrücke_der_A44.jpg); b) footbridge in Warsaw (photograph by the first author).

application of the double split-pylon with horizontal reinforcing member allowed to reduce pylon height to 81 m, with similar material usage and total cost of the bridge as in classical design. A smaller-scale example of split-pylon design can be found in the footbridge at Żwirki and Wigury Street in Warsaw, which is supported by central biforked pylon reinforced by steel cables (Fig. 14b).

6. Conclusions

Michell structures are nowadays a known concept to architects and structural designers. Simplified versions of optimal Michell's layouts are used to develop innovative designs of high-rise buildings, large-scale roofs and long-span bridges. In the case of buildings, Michell's layouts are used to propose novel structural systems resisting substantial bending forces caused by wind loads. In the case of large-scale roofs, Michell's theory is often applied together with elements of the tensegrity principle and it results in the interplay of tensile and compressive forces. Eventually, in the case of bridges the similarity to Michell structures is reflected in the application of the arch bridge and suspended bridge as optimal structural forms, as well as in the application of atypical split-pylon form of cable-stayed bridge. More detailed and further information on applications of the theory of Michell structures in civil and mechanical engineering can be found in the book [16].

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